Statistical Analysis of the CAPM

II. Black CAPM

Brief Review of the Black CAPM

- The Black CAPM assumes that

  (i) all investors act according to the $\mu - \sigma$ rule,
  (ii) face no short selling constraints, and
  (iii) exhibit perfect agreement with respect to the probability distribution of asset returns.

- It is not assumed that they can lend and borrow at a common risk–free rate.

- Under these assumptions, the market portfolio is a mean–variance efficient portfolio.

- Thus, there is a portfolio Z, i.e., the *zero–beta portfolio with respect to the market portfolio*, such that for each risky asset or portfolio of risky assets i, we have

$$\mu_i = \mu_z + \beta_i(\mu_m - \mu_z), \tag{1}$$

  where μ_m is the expected return of the market portfolio.

Framework for Estimation and Testing

- The CAPM relationship (1) is expressed in terms of expected values, which are not observable.

- To obtain a model with observable quantities, we describe returns using the following *market model*:

$$r_{it} = \alpha_i + \beta_i r_{m,t} + \epsilon_{it} \quad i = 1, \ldots, N \tag{2}$$

$$E(\epsilon_{it}) = 0, \quad i = 1, \ldots, N \tag{3}$$

$$E(\epsilon_{it}\epsilon_{jt'}) = \begin{cases} \sigma_{ij} & \text{if } t = t' \\ 0 & \text{if } t \neq t' \end{cases} \quad i, j = 1, \ldots, N \tag{4}$$

$$E(r_{m,t}\epsilon_{i,t}) = 0, \quad i = 1, \ldots, N. \tag{5}$$

- Here $r_{i,t}$ is the return of asset i in period t, and $r_{m,t}$ is the return of the market portfolio in period t.

- This is very similar to the framework employed for testing the Sharpe–Lintner CAPM.

- However, in contrast to the market model considered last week, the relation (2) is *not* stated in terms of excess returns.

- According to equation (4), the asset–specific error terms may be correlated.

- Thus, we allow for a non-diagonal covariance matrix, $\mathbf{\Sigma}$, of the vector $\boldsymbol{\epsilon}_t = [\epsilon_{1t}, \ldots, \epsilon_{Nt}]'$,

$$COV(\boldsymbol{\epsilon}_t) = \mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{bmatrix}$$

- Conditional on the excess return of the market, we then also have

$$COV(\boldsymbol{r}_t) = \mathbf{\Sigma}, \qquad (6)$$

where $\boldsymbol{r}_t = [r_{1t}, \ldots, r_{2t}]'$.

- We will also assume that the error terms follow a multivariate normal distribution, i.e.,

$$\boldsymbol{\epsilon}_t \overset{iid}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}). \tag{7}$$

- The Black CAPM implies a restriction on the intercept terms in (2), namely,

$$\alpha_i = (1 - \beta_i)\mu_z, \quad i = 1, \ldots, N, \tag{8}$$

or, using vector notation,

$$\boldsymbol{\alpha} = (\mathbf{1}_N - \boldsymbol{\beta})\mu_z. \tag{9}$$

- Equation (9) imposes a nonlinear restriction on the parameters, because μ_z is not known and has to be estimated, along with the further unknown parameters of the (restricted) model, i.e., $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$.

Estimation of the Parameters

- Write the market model as

$$\begin{aligned} \boldsymbol{r}_t &= \boldsymbol{\alpha} + \boldsymbol{\beta} r_{m,t} + \boldsymbol{\epsilon}_t, \quad t = 1, \ldots, T, \\ \boldsymbol{\epsilon}_t &\overset{iid}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}), \end{aligned}$$

where $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_N]'$, and $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_N]'$.

- The maximum likelihood estimator (MLE) for the unconstrained model has been derived last week, and is given by

$$\widehat{\boldsymbol{\alpha}} = \bar{\boldsymbol{r}} - \widehat{\boldsymbol{\beta}} \bar{r}_m, \tag{10}$$

$$\begin{aligned} \widehat{\boldsymbol{\beta}} &= \frac{\sum_{t=1}^{T} (\boldsymbol{r}_t - \bar{\boldsymbol{r}})(r_{m,t} - \bar{r}_m)}{\sum_{t=1}^{T} (r_{m,t} - \bar{r}_m)^2} \qquad (11) \\ &= \frac{\sum_{t=1}^{T} (\boldsymbol{r}_t - \bar{\boldsymbol{r}})(r_{m,t} - \bar{r}_m)}{T \widehat{\sigma}_m^2} \\ &= \frac{\sum_{t=1}^{T} (r_{m,t} - \bar{r}_m) \boldsymbol{r}_t}{T \widehat{\sigma}_m^2}, \end{aligned}$$

and

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}\widehat{\boldsymbol{\epsilon}}_t\widehat{\boldsymbol{\epsilon}}_t' \tag{12}$$

$$= \frac{1}{T}\sum_{t=1}^{T}(\boldsymbol{r}_t - \widehat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\beta}}r_{m,t})(\boldsymbol{r}_t - \widehat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\beta}}r_{m,t})'.$$

where

$$\bar{\boldsymbol{r}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{r}_t, \quad \bar{r}_m = \frac{1}{T}\sum_{t=1}^{T}r_{m,t},$$

$$\widehat{\sigma}_m^2 = \frac{1}{T}\sum_{t=1}^{T}(r_{m,t} - \bar{r}_m)^2.$$

Estimation of the Restricted Model

- Recall that the Black CAPM imposes

$$\alpha = (1_N - \beta)\mu_z. \tag{13}$$

- The parameters to estimate are μ_z, β and Σ, and the log–likelihood function is

$$\begin{aligned}\log L(\mu_z, \beta, \Sigma) &= -\frac{NT}{2}\log(2\pi) - \frac{T}{2}\log|\Sigma| \\ &\quad -\frac{1}{2}\sum_{t=1}^{T}\widehat{\epsilon}_t'\Sigma^{-1}\widehat{\epsilon}_t,\end{aligned}$$

where

$$\widehat{\epsilon}_t = r_t - (1_N - \widehat{\beta})\widehat{\mu}_z - \widehat{\beta} r_{m,t}. \tag{14}$$

- Note that, by the same arguments as last week, whatever the estimators of β and μ_z will be, the estimator of $\mathbf{\Sigma}$ is

$$\begin{aligned}\widehat{\mathbf{\Sigma}}(\widehat{\mu}_z, \widehat{\beta}) &= \frac{1}{T}\sum_{t=1}^{T}\widehat{\boldsymbol{\epsilon}}_t\widehat{\boldsymbol{\epsilon}}_t' \qquad (15)\\ &= \frac{1}{T}\sum_{t=1}^{T}(\boldsymbol{r}_t - (1_N - \widehat{\beta})\widehat{\mu}_z - \widehat{\boldsymbol{\beta}}r_{m,t})\\ &\quad \times(\boldsymbol{r}_t - (1_N - \widehat{\beta})\widehat{\mu}_z - \widehat{\boldsymbol{\beta}}r_{m,t})'.\end{aligned}$$

- Moreover, for any given $\widehat{\mu}_z$, $\widehat{\beta}$ will be the equation–by–equation OLS estimator of the regression through the origin

$$(r_t - 1_N\widehat{\mu}_z) = \beta(r_{m,t} - \widehat{\mu}_z), \quad t = 1, \ldots, T.$$

It follows that

$$\widehat{\beta}(\widehat{\mu}_z) = \frac{\sum_{t=1}^{T}(r_t - 1_N\widehat{\mu}_z)(r_{m,t} - \widehat{\mu}_z)}{\sum_{t=1}^{T}(r_{m,t} - \widehat{\mu}_z)^2}. \qquad (16)$$

- From last week's analysis, we also know that the log–likelihood function, evaluated at the MLE, is

$$\log L = -\frac{NT}{2}[\log(2\pi) + 1] - \frac{T}{2}\log|\widehat{\Sigma}|.$$

- Thus, we have to find $\widehat{\beta}$ and $\widehat{\mu}_z$ such that

$$\log|\widehat{\Sigma}| = \log\left|\frac{1}{T}\sum_{t=1}^{T}(r_t - \widehat{\mu}_z(1_N - \widehat{\beta}) - \widehat{\beta}r_{m,t}) \times (r_t - \widehat{\mu}_z(1_N - \widehat{\beta}) - \widehat{\beta}r_{m,t})'\right| \qquad (17)$$

is minimized.

- But, as we have seen in (16), $\widehat{\beta}$ can be written as a function of $\widehat{\mu}_z$, namely

$$\widehat{\beta}(\widehat{\mu}_z) = \frac{\sum_{t=1}^{T} (r_t - 1_N \widehat{\mu}_z)(r_{m,t} - \widehat{\mu}_z)}{\sum_{t=1}^{T} (r_{m,t} - \widehat{\mu}_z)^2}. \tag{18}$$

- Thus, (17) can be written as a function of just a single variable, $\widehat{\mu}_z$.

- Hence, we can find the MLE of the restricted model by first identifying $\widehat{\mu}_z$.

- This can be done, for example, by conducting a simple grid–search.

- Then compute $\widehat{\beta}$ via (18) and finally evaluate $\widehat{\Sigma}$ using equation (15).

Likelihood Ratio (LR) Test

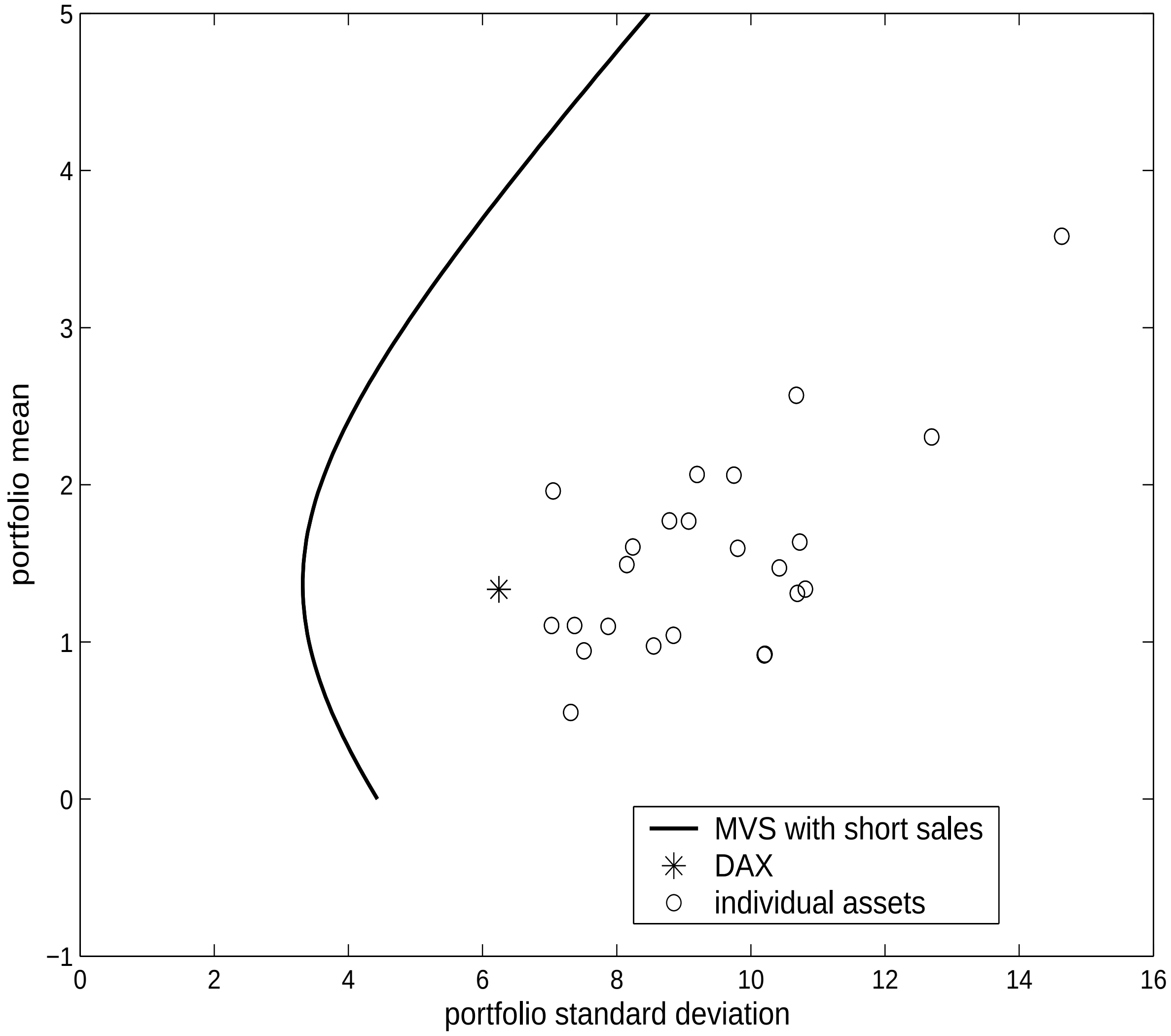
- Having estimated the parameters of both the unrestricted as well as those of the restricted market model, we can conduct a likelihood ratio (LR) test.

- If
  - $\widehat{\Sigma}_1$ denotes the estimated error term covariance matrix under the unrestricted model, and
  - $\widehat{\Sigma}_0$ is the estimated error term covariance matrix under the null hypothesis,

  then, by the same arguments as last week, the LR test statistic is

$$\mathcal{LR} = T \left[\log |\widehat{\Sigma}_0| - \log |\widehat{\Sigma}_1| \right] \overset{asy}{\sim} \chi^2(N-1).$$

- Note that the degrees of freedom of the null distribution is $N-1$. Relative to the Sharpe–Lintner model, we lose one degree of freedom because μ_z is a free parameter.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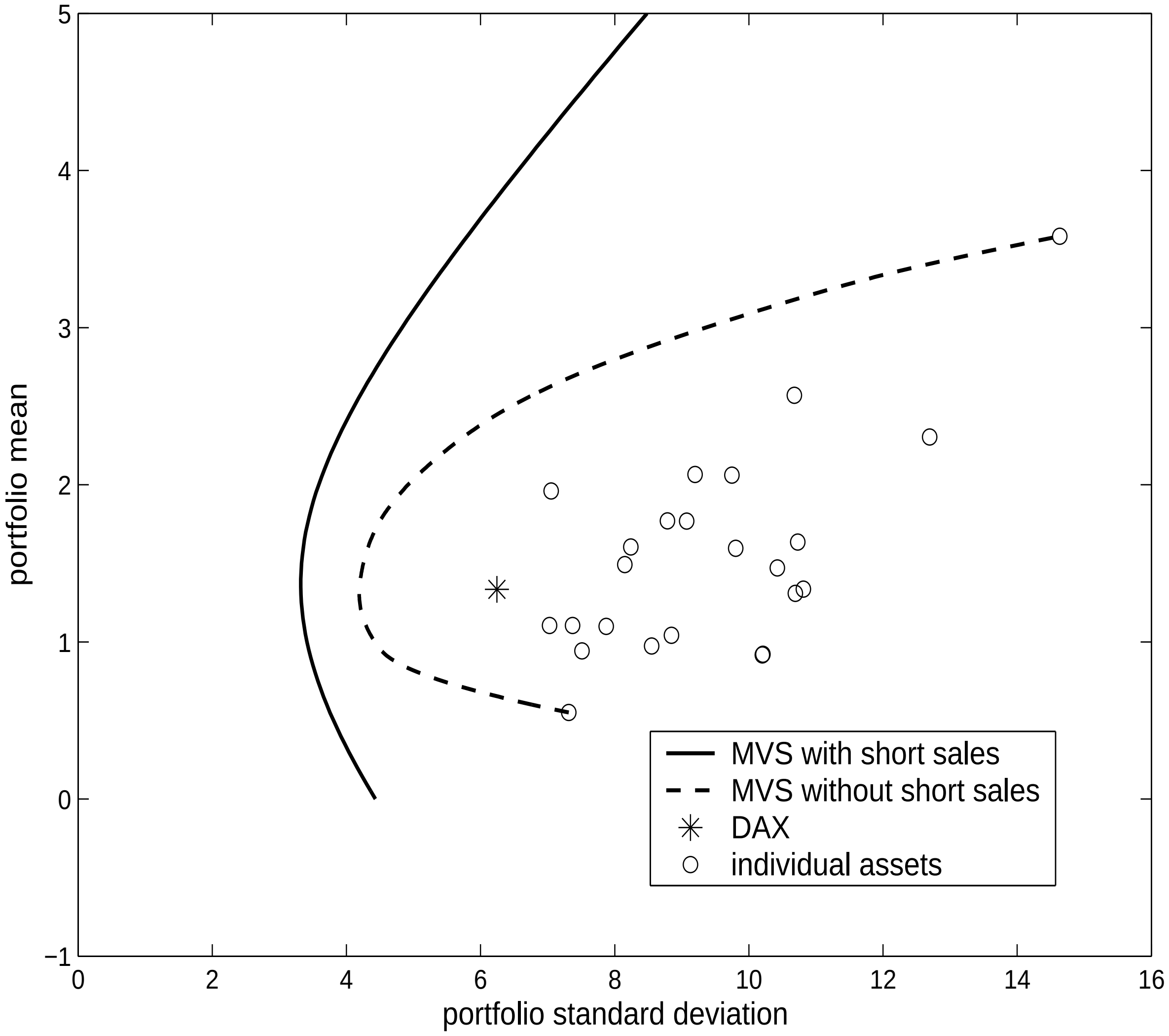
DAX
individual assets



portfolio mean
portfolio standard deviation
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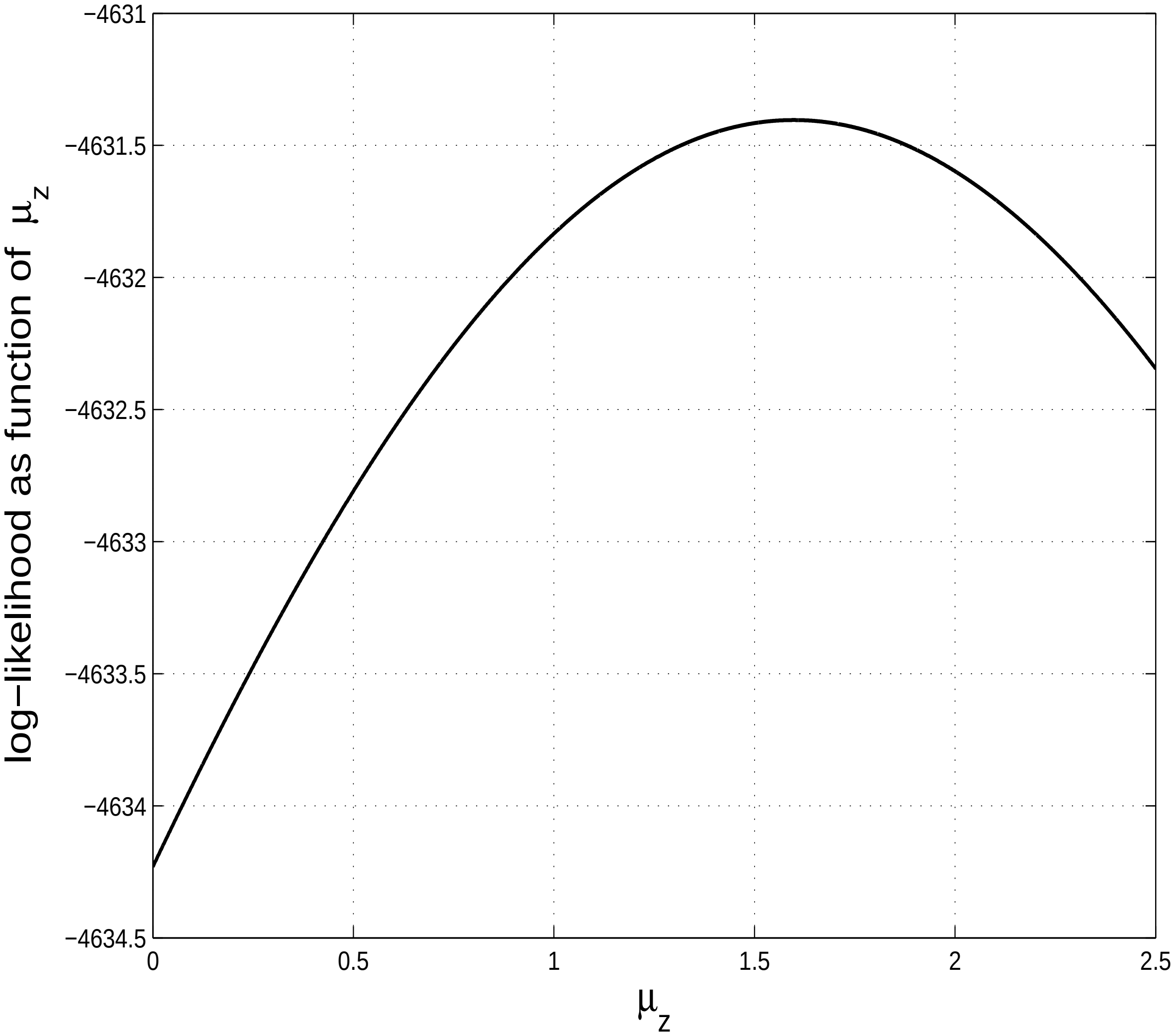

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log−likelihood as function of μz
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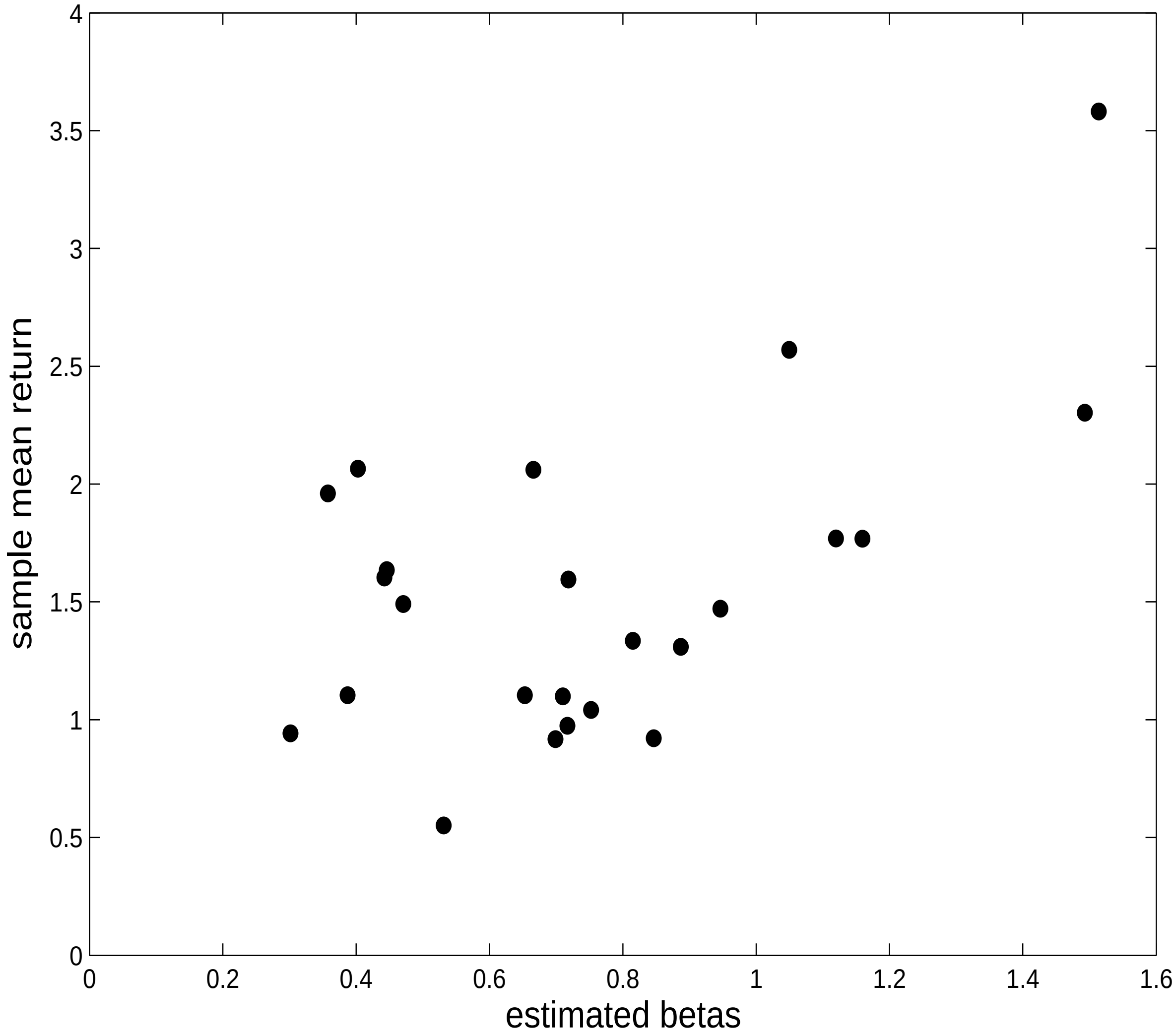

- We have also seen that the asymptotic likelihood ratio test may exhibit poor performance in finite samples.

- To mitigate these effects, the adjusted statistic

$$\mathcal{LR}^{\star} = \left(T - \frac{N}{2} - 2\right)\left[\log|\widehat{\Sigma}_0| - \log|\widehat{\Sigma}_1|\right] \overset{asy}{\sim} \chi^2(N-1)$$

  has been shown to more closely match the χ^2 distribution in finite samples.

- There also exists a further device that provides a useful check.[1]

- This also gives rise to a closed–from estimator for the zero-beta rate, μ_Z.

[1] Cf. Shanken, J. (1986) Testing Portfolio Efficiency when the Zero–Beta Rate in Unknown: A Note. *Journal of Finance 41*, 269-276

Lower Bound for the Exact Distribution

- Suppose for the moment that μ_z is known.

- Then, we can proceed as last week when testing the Sharpe–Linter model, i.e., we can consider the "excess return" market model

$$r_t - \mu_z 1_N = \alpha + \beta(r_{m,t} - \mu_z) + \epsilon_t. \tag{19}$$

- The zero–beta CAPM is true if $\alpha = 0$.

- The estimates of the unrestricted model are

$$\begin{aligned}
\widehat{\alpha}_1 &= \bar{r} - 1_N \mu_z - \widehat{\beta}(\bar{r}_m - \mu_z) \\
\widehat{\beta}_1 &= \frac{\sum_t (r_t - \bar{r})(r_{m,t} - \bar{r}_{m,t})}{\sum_t (r_{m,t} - \bar{r}_m)^2} \\
\widehat{\Sigma}_1 &= \frac{1}{T} \sum_t [r_t - \bar{r} - \widehat{\beta}_1(r_{m,t} - \bar{r}_m)] \\
&\quad \times [r_t - \bar{r} - \widehat{\beta}_1(r_{m,t} - \bar{r}_m)]'.
\end{aligned}$$

Note that $\widehat{\beta}_1$ and $\widehat{\Sigma}_1$ do not depend on μ_z, and, thus, the value of the log–likelihood function does also not depend on μ_z, as, at the maximum,

$$\log L_1 = -\frac{NT}{2}[\log(2\pi) + 1] - \frac{T}{2}\log|\widehat{\Sigma}_1|.$$

- The MLE under the restriction that $\alpha = 0$ is

$$\begin{aligned} \widehat{\beta}_0(\mu_z) &= \frac{\sum_t (r_t - 1_N\mu_z)(r_{m,t} - \mu_z)}{\sum_t (r_{m,t} - \mu_z)^2} \qquad (20) \\ \widehat{\Sigma}_0(\mu_z) &= \frac{1}{T}\sum_t (r_t - \mu_z(1_N - \widehat{\beta}_0) - \widehat{\beta}_0 r_{m,t}) \\ &\quad \times (r_t - \mu_z(1_N - \widehat{\beta}_0) - \widehat{\beta}_0 r_{m,t})'. \end{aligned}$$

- The value of the constrained log–likelihood is

$$\log L_0(\mu_z) = -\frac{NT}{2}[\log(2\pi)+1] - \frac{T}{2}\log|\widehat{\Sigma}_0(\mu_z)|,$$

which can be viewed as a function of only one variable, i.e., μ_z.

- Consequently, the likelihood ratio test statistic,

$$\mathcal{LR}(\mu_z) = T\left[\log|\widehat{\Sigma}_0(\mu_z)| - \log|\widehat{\Sigma}_1|\right], \qquad (21)$$

can be viwed as a function of only μ_z.

- Obviously, the value of μ_z which minimizes the likelihood ratio statistic will be the MLE of μ_z.

- (Recall that $|\widehat{\Sigma}_1|$ does not depend on μ_Z.)

- In the last week, we have developed a relation between the LR test and the F test for the Sharpe–Linter CAPM, which used a formula expressing $|\widehat{\Sigma_0}|$ in terms of $|\widehat{\Sigma_1}|$ and $\widehat{\alpha}$.

- Repeating the same line of arguments shows that (21) can be written as

$$\mathcal{LR}(\mu_z) = T \log \left[\widehat{\alpha}' \widehat{\Sigma}_1^{-1} \widehat{\alpha} \frac{\widehat{\sigma}_m^2}{(\bar{r}_m - \mu_z)^2 + \widehat{\sigma}_m^2} + 1 \right], \tag{22}$$

where

$$\widehat{\alpha} = (\bar{r} - \widehat{\beta}_1 \bar{r}_m) - (1_N - \widehat{\beta}_1)\mu_z. \tag{23}$$

is a function of μ_z

- Thus, the MLE of μ_z is the value which minimizes

$$
\begin{aligned}
g(\mu_z) &= \widehat{\alpha}'\widehat{\Sigma}_1^{-1}\widehat{\alpha}\frac{\widehat{\sigma}_m^2}{(\bar{r}_m - \mu_z)^2 + \widehat{\sigma}_m^2} \\
&= \frac{[\mu_z^2 a - 2b\mu_z + c]\widehat{\sigma}_m^2}{\widehat{\sigma}_m^2 + (\bar{r}_m - \mu_z)^2},
\end{aligned}
$$

where

$$
\begin{aligned}
a &= (1_N - \widehat{\beta}_1)'\widehat{\Sigma}_1^{-1}(1_N - \widehat{\beta}_1), \\
b &= (1_N - \widehat{\beta}_1)'\widehat{\Sigma}_1^{-1}(\bar{r} - \widehat{\beta}_1\bar{r}_m), \\
c &= (\bar{r} - \widehat{\beta}_1\bar{r}_m)'\widehat{\Sigma}_1^{-1}(\bar{r} - \widehat{\beta}_1\bar{r}_m).
\end{aligned}
$$

- It follows that we can find the MLE of μ_z by solving

$$\frac{dg(\mu_z)}{d\mu_z} = \frac{(2a\mu_z - 2b)[(\bar{r}_m - \mu_z)^2 + \widehat{\sigma}_m^2]}{[(\bar{r}_m - \mu_z)^2 + \widehat{\sigma}_m^2]^2} - \frac{(\mu_z^2 a - 2b\mu_z + c)2(\mu_z - \bar{r}_m)}{[(\bar{r}_m - \mu_z)^2 + \widehat{\sigma}_m^2]^2} = 0,$$

that is, μ_z is a root of the quadratic

$$A\mu_z^2 + B\mu_z + C = 0, \qquad (24)$$

where

$$\begin{aligned} A &= b - a\bar{r}_m, \\ B &= a(\widehat{\sigma}_m^2 + \bar{r}_m^2) - c, \\ C &= -b(\widehat{\sigma}_m^2 + \bar{r}_m^2) + c\bar{r}_m. \end{aligned}$$

- This is a closed–form solution for $\widehat{\mu}_z$.

- It can be shown that the roots of (24) are real.

- The maximum likelihood estimator, then, is the root which corresponds to the smaller value of $\log(\det(\widehat{\Sigma}(\mu_z)))$.

- Once $\widehat{\mu}_z$ is determined, we can apply the closed form expressions (20) to determine the MLE for the other parameters.

- Now let us return to our objective of finding a test which does not rely on asymptotic arguments.

- Recall that, with μ_z known, we could use the statistic

$$J(\mu_z) = \frac{T-N-1}{N}\left[1+\frac{(\bar{r}_m-\mu_z)^2}{\widehat{\sigma}_m^2}\right]^{-1}\widehat{\alpha}'\widehat{\Sigma}_1^{-1}\widehat{\alpha} \tag{25}$$

  to conduct an exact finite–sample test.

- This uses the result that, in this case, (25) has an F distribution with N degrees of freedom in the numerator and $T-N-1$ degrees of freedom in the denominator.

- As μ_z is not known, this cannot be done.

- The MLE of μ_z, $\widehat{\mu}_z$, is the value which minimizes the LR test statistic, $\mathcal{LR}(\mu_z)$.

- As we know that $J(\mu_z)$ is a monotonic transformation of $\mathcal{LR}(\mu_z)$, $\widehat{\mu}_z$ also minimizes $J(\mu_z)$.

- Thus,
$$J(\widehat{\mu}_z) \le J(\mu_z), \tag{26}$$
where μ_z is the true value of the zero–beta portfolio's mean.

- That is, the F test based on $\widehat{\mu}_z$ and using the "exact" F distribution will accept too often.

- But we know that, if it *rejects*, it will reject for *any* value of μ_z, and we need not resort to asymptotic approximations in this case.

- This is a useful check because we have seen that the asymptotic likelihood ratio test rejects too often.