The Value-at-Risk (VaR) concept has emerged as one of the most prominent measures of downside market risk.

The importance of this concept is mainly due to its current use in the regulation of financial institutions by supervising authorities.
Definition of VaR

- VaR for a portfolio can be defined in terms of a quantile of the portfolio’s profit/loss distribution for a given horizon (typically a day or a week) and a given shortfall probability (typically chosen between 1% and 5%).

- For example, the Value-at-Risk with shortfall probability 1% is just the (negative\(^1\) of the) 0.01–quantile of the portfolio’s profit/loss distribution.

- The portfolio’s profit/loss distribution can easily be deduced from the return distribution. For an investment of 1$, the profit/loss distribution is just the return distribution.

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\(^1\)The negative sign is due to the convention of reporting VaR as a positive number.
Thus, the VaR with shortfall probability $\alpha$, $VaR(\alpha)$, can be defined as

$$\Pr(\Delta P_t < -VaR(\alpha)) = \alpha,$$  \hspace{2cm} (1)

where $P_t$ is the portfolio’s value at time $t$, and $\Delta P_t = P_t - P_{t-1}$ is the profit or loss between time $t - 1$ and time $t$.

The negative sign in (1) is due to the convention of reporting VaR as a positive number, i.e., loss.

Note that the percentage return

$$r_t = 100 \times \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{\Delta P_t}{P_{t-1}}$$

$$\implies \Delta P_t = \frac{r_t}{100} P_{t-1},$$

so that the Profit/Loss distribution, i.e., the distribution of $\Delta P_t$, can be derived from the return distribution, i.e., the distribution of $r_t$. 
Thus, economically, the VaR measure can be defined to be a loss large enough so that the probability that the portfolio could post a larger loss is at most 1% (or, in general, $100 \times \alpha\%$).
Simple Example

- Suppose we construct a portfolio from two risky assets, with returns $r_1$ and $r_2$. We assume that these have a bivariate normal distribution with mean and covariance matrix

$$
\mu = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 9 & 3 \\ 3 & 4 \end{bmatrix},
$$

respectively.

- Our portfolio has weights $x = [0.5, 0.5]'$. Due to the normality assumption, we know the exact portfolio distribution, which is also normal with mean and variance

$$
x'\mu = 2.5, \quad \text{and} \quad x'\Sigma x = 4.75,
$$

respectively.
The $\alpha$–quantile of the normal distribution with mean $\mu$ and variance $\sigma^2$, $z_\alpha(\mu, \sigma^2)$, is readily calculated. From

$$\alpha = \int_{-\infty}^{z_\alpha(\mu, \sigma^2)} \phi(x; \mu, \sigma^2) dx$$

$$=: \Phi(z_\alpha(\mu, \sigma^2); \mu, \sigma^2)$$

$$= \Phi \left( \frac{z_\alpha(\mu, \sigma^2) - \mu}{\sigma} ; 0, 1 \right)$$

$$=: \Phi \left( \frac{z_\alpha(\mu, \sigma^2) - \mu}{\sigma} \right) ,$$

where $\phi(\cdot; \mu, \sigma^2)$ and $\Phi(\cdot; \mu, \sigma^2)$ are the normal pdf and cdf with mean $\mu$ and variance $\sigma^2$, respectively, we get

$$z_\alpha(\mu, \sigma^2) = \mu + z_\alpha \sigma ,$$

where $z_\alpha$ is the $\alpha$–quantile of the standard normal distribution.
• The 0.01–quantile of the standard normal is $z_{0.01} = -2.3263$.

• Thus, the 0.01–quantile of our portfolio return distribution is

$$z_{0.01}(2.5, 4.75) = -2.3263 \sqrt{4.75} + 2.5$$
$$= -2.5702.$$ 

• Suppose, we have invested 25000$ in the portfolio. Then, the VaR with shortfall probability of 1% of this portfolio is

$$VaR(0.01) = \frac{2.5702}{100} \times 25000\$ = 642.54\$,$$

that is, with probability 0.99, our loss will not exceed 642.54 $. 


Value-at-Risk and Down-Side Risk Portfolio Choice

• Suppose a portfolio manager faces a Value-at-Risk constraint, i.e., the Value-at-Risk with shortfall probability $\alpha$ must not exceed a given level.

• Then, as $r_p = 100 \times \Delta P/P$ and with $\tau = -100 \times \text{VaR}_{(\alpha)}/P$, the constraint can be written

$$P(r_p \leq \tau) \leq \alpha.$$  \hfill (2)

• This coincides with the shortfall constraint, so that the criteria for portfolio selection based on down-side risk can be applied by a portfolio manager facing VaR constraints.