Exercise Course “Portfolio Analysis”

Problem Set 1

Problem 1

a) A person with initial wealth \( w \) has an expected utility function of the form \( U(W) = \log W \). She is offered the opportunity to bet on the flip of a coin that has a probability \( p \) of coming up heads. If she bets \( x \), she will have \( w + x \) if head comes up and \( w - x \) if tails comes up. Solve for the optimal \( x (x \geq 0) \) as a function of \( p \). What if \( p = 0.5 \)?

Does the investor of this example exhibit decreasing absolute risk aversion?

b) A person has an expected utility function of the form \( U(W) = \sqrt{W} \). He initially has wealth of 4. He also has a lottery ticket that will be worth 12 with probability 0.5 and will be worth 0 (nothing) with probability 0.5. What is the lowest price, \( p \), at which he would sell the ticket?

Problem 2

a) A person with initial wealth \( W_0 \) and expected utility function \( U(W) = \log W \) has two investment alternatives: A risk-free asset, which pays no interest (e.g., money), and a risky asset, the return \( (R) \) on which will be equal to \( r_1 < 0 \) with probability \( p \) and equal to \( r_2 > 0 \) with probability \( 1 - p \). Denote the fraction of initial wealth to be invested in the risky asset by \( x \), and the return on the portfolio formed from the risk-free and the risky asset by \( r_p \). Find the \( x \) which maximizes the expected utility of wealth in period 1, \( W_1 = (1 + r_p)W_0 \). Denote this solution by \( x^* \). What is the condition for \( x^* > 0 \)?

b) Now consider a problem similar to that in a), but the expected utility function of the investor is \( U(W) = -\exp(-cW) \), \( c > 0 \), and the return of the risky asset is normally distributed with mean \( \mu \) and variance \( \sigma^2 \).
Problem 3: Properties and interpretation of correlation

a) (i) Prove the Cauchy–Schwarz Inequality: For two random variables $V$ and $W$ with finite second moments, \( \{E(VW)\}^2 \leq E(V^2)E(W^2) \). (Hint: Consider the function \( g(\lambda) = E((V - \lambda W)^2) \geq 0 \) and choose \( \lambda \) appropriately.)

(ii) Conclude that for random variables $X$ and $Y$, $-1 \leq \rho_{XY} \leq 1$, where $\rho_{XY}$ is the coefficient of correlation between $X$ and $Y$.

b) Correlation is a measure for the strength of linear association between two random variables. Consider random variables $X$ and $Y$ with means $\mu_X$ and $\mu_Y$, variances $\sigma_X^2$ and $\sigma_Y^2$, covariance $\sigma_{XY}$, and correlation $\rho_{XY}$. Define the random variable $\hat{Y} = a + bX$ to represent the predictions of $Y$ based on a linear function of $X$.

(i) Show that, if $a$ and $b$ are chosen to minimize the expected squared distance, \[
E[d^2(Y, \hat{Y})] = E[(Y - \hat{Y})^2] = E[(Y - (a + bX))^2],
\]
the result is $a = \mu_Y - (\sigma_{XY}/\sigma_X^2)\mu_X$, and $b = \sigma_{XY}/\sigma_X^2$.

(ii) Define $V = Y - \hat{Y}$ to represent the deviations between outcomes of $Y$ and outcomes of the best linear prediction of $Y$; i.e., $\hat{Y}$ as determined by the minimization of (1). Show that \[
\sigma_Y^2 = \sigma_{\hat{Y}}^2 + \sigma_V^2,
\]
where \[
\sigma_{\hat{Y}}^2 = E[(\hat{Y} - E(\hat{Y}))^2] = \rho_{XY}^2 \sigma_Y^2,
\]
\[
\sigma_V^2 = E(V^2) = (1 - \rho_{XY}^2)\sigma_Y^2.
\]
Interpret this result.
c) The property of being positively correlated is not transitive. That is, if \( X, Y \) and \( Z \) are random variables and \( X \) and \( Y \) are positively correlated and \( Y \) and \( Z \) are likewise positively correlated, it is not necessarily the case that \( X \) and \( Z \) are also positively correlated. To illustrate, show that:

Let \( U, V \) and \( W \) be any nontrivial independent random variables with zero mean, and define \( X, Y \) and \( Z \) by \( X = U + V, Y = W + V, \) and \( Z = W - U \). Then \( \rho_{XY} \) and \( \rho_{YZ} \) are positive and \( \rho_{XZ} \) is negative, where \( \rho \) denotes correlation.\(^1\)

\(^1\)For further analysis, see Langford, Schwertman and Owens (2001): Is the Property of Being Positively Correlated Transitive? The American Statistician 55, 322-325.