MVS with risky assets only

The minimum variance set (MVS) is given by

\[
\sigma_p^2 = \frac{d}{c} \left( \mu_p - \frac{b}{c} \right)^2 + \frac{1}{c}
\]

\[
= \frac{d}{c} (\mu_p - \mu_{GMVP})^2 + \sigma_{GMVP}^2,
\]

where the global minimum variance portfolio (GMVP) has

\[
\mu_{GMVP} = \frac{b}{c}
\]

\[
\sigma_{GMVP}^2 = \frac{1}{c}.
\]

In \((\sigma_p, \mu_p)\) space, we have the hyperbola

\[
\mu_p = \frac{b}{c} \pm \sqrt{\frac{d}{c} \left( \sigma_p^2 - \frac{1}{c} \right)}
\]

\[
= \mu_{GMVP} \pm \sqrt{\frac{d}{c} \left( \sigma_p^2 - \sigma_{GMVP}^2 \right)}.
\]
The efficient set (ES) is just

$$\mu_{ES}^p = \frac{b}{c} + \sqrt{\frac{d}{c} \left( \sigma_p^2 - \frac{1}{c} \right)}$$

$$= \mu_{GMVP} + \sqrt{\frac{d}{c} \left( \sigma_p^2 - \sigma_{GMVP}^2 \right)}, \quad (3)$$

i.e., the upper limb of the minimum–variance hyperbola, since these portfolios maximize expected return, given their return variances.

The slope of the MVS is

$$\frac{d\mu_p}{d\sigma_p} = \pm \sqrt{\frac{d}{c}} \frac{2\sigma_p}{2\sqrt{\sigma_p^2 - \sigma_{GMVP}^2}} = \pm \sqrt{\frac{d}{c}} \frac{1}{\sqrt{1 - \sigma_{GMVP}^2/\sigma_p^2}}, \quad (4)$$

from which the slope of the asymptotes of the minimum–variance hyperbola is

$$\lim_{\sigma_p \to \infty} \frac{d\mu_p}{d\sigma_p} = \pm \sqrt{\frac{d}{c}}, \quad (5)$$
The asymptotes of the hyperbola are thus

\[ L(\sigma_p) = \frac{b}{c} \pm \sqrt{\frac{d}{c}} \sigma_p. \]

(6)

Note that

\[
\left( \sqrt{\sigma_p^2 - \sigma_{GMVP}^2} - \sqrt{\sigma_p^2} \right) \left( \sqrt{\sigma_p^2 - \sigma_{GMVP}^2} + \sqrt{\sigma_p^2} \right) = \left( \sigma_p^2 - \sigma_{GMVP}^2 \right) - \sigma_p^2 = -\sigma_{GMVP}^2,
\]
i.e.,

\[
\sqrt{\sigma_p^2 - \sigma_{GMVP}^2} - \sqrt{\sigma_p^2} = \frac{-\sigma_{GMVP}^2}{\sqrt{\sigma_p^2 - \sigma_{GMVP}^2} + \sqrt{\sigma_p^2}},
\]
and so

\[
\lim_{\sigma_p \to \infty} \left( \frac{d}{c} \right) \left( \sqrt{\sigma_p^2 - \sigma_{GMVP}^2} - \sqrt{\sigma_p^2} \right) = \lim_{\sigma_p \to \infty} \sqrt{\frac{d}{c} \frac{\sigma_{GMVP}^2}{\sqrt{\sigma_p^2 - \sigma_{GMVP}^2} + \sqrt{\sigma_p^2}}} = 0,
\]
i.e., as $\sigma_p$ grows, the (upper and lower limb of the) MVS–hyperbola approaches the asymptotes (6), i.e., asymptotically, the upper and lower limb of the MVS become straight lines with slope $\pm \sqrt{d/c}$ (cf. Figure).

Investors will choose efficient portfolios, i.e., portfolios in the efficient set (upper limb), since these also maximize return for a prespecified variance.

The trade–off between risk and expected return for efficient portfolios is apparent.
\[ \frac{b}{c} \pm \sqrt{\frac{d}{c}} \sigma_p \]

\[ \frac{b}{c} \pm \sqrt{\frac{d}{c} \left( \sigma_p^2 - \frac{1}{c} \right)} \]
Zero–beta portfolios

- For two MVS portfolios $p$, and $q$, we have, since $x_p = \Sigma^{-1} R \Phi^{-1} \tilde{\mu}_p$, $\Phi = R' \Sigma^{-1} R$,

\[
\text{Cov}(R_p, R_q) = x'_p \Sigma x_q = \tilde{\mu}'_p \Phi^{-1} R' \Sigma^{-1} \Sigma \Sigma^{-1} R \Phi^{-1} \tilde{\mu}_q
\]

\[
= \tilde{\mu}'_p \Phi^{-1} \tilde{\mu}_q = \frac{c \mu_p \mu_q - b(\mu_p + \mu_q) + a}{d}
\]

\[
= \frac{c}{d} \left( \mu_p - \frac{b}{c} \right) \left( \mu_q - \frac{b}{c} \right) + \frac{1}{c}. \quad (7)
\]

- Note that $b/c = \mu_{GMVP}$. Hence, the equation implies that for each MVS portfolio $p$, except the GMVP, there is a unique MVS portfolio which is uncorrelated with portfolio $p$. (This is the zero–beta portfolio with respect to $p$ and is denoted by $Z$.)
Since $1/c > 0$, the product (first term in) (7) must be negative for the covariance to become zero, so that the zero–beta portfolio is always on the opposite branch of the hyperbola, i.e., if $p$ is on the upper limb, then $Z$ is on the lower limb and vice versa.

The mean $\mu_Z$ of the zero–beta portfolio with respect to $p$ can be obtained by setting (7) to zero and solving,

$$\mu_Z = \frac{b\mu_p - a}{c\mu_p - b}. \quad (8)$$

We can also show that the tangent to the MVS at portfolio $p$ intersects the return axis at $\mu_Z$. The associated equation is

$$\mu_p = \mu_Z + \frac{d\mu}{d\sigma}_{MV S}^{MVS} \times \sigma_p. \quad (9)$$
To see this, note that, on the MVS, where \( \sigma^2 = (c\mu^2 - 2b\mu + a)/d \), \( d = ac - b^2 \),

\[
\left. \frac{d\mu}{d\sigma} \right|_{MVS} = \pm \sqrt{\frac{d}{c}} \frac{\sigma}{\sqrt{\sigma^2 - 1/c}}
\]

\[
= \pm \frac{d\sigma}{\sqrt{d\sigma^2 - d}}
\]

\[
= \pm \frac{d\sigma}{\sqrt{c^2\mu^2 - 2bc\mu + ac - d}}
\]

\[
= \pm \frac{d\sigma}{\sqrt{(c\mu - b)^2}}
\]

\[
= \frac{d\sigma}{c\mu - b}.
\]
• Hence

\[ \mu_p - \frac{d\mu}{d\sigma}\bigg|_{\sigma=\sigma_p}^{\text{MV S}} \times \sigma_p \]

\[ = \mu_p - \frac{d\sigma_p}{c\mu_p - b} \times \sigma_p \]

\[ = \mu_p - \frac{c\mu_p^2 - 2b\mu_p + a}{c\mu_p - b} \]

\[ = \frac{b\mu_p - a}{c\mu_p - b} \]

\[ = \mu_Z, \]

which proves (9).

• Finally, the following equation is the basis for the CAPM:

\[ \mu_i = \mu_Z + \beta_i(\mu_p - \mu_Z). \quad (10) \]

where \( \mu_i \) is the expected return of asset \( i \), which may also be a portfolio of assets, and

\[ \beta_i = \frac{\text{Cov}(R_i, R_p)}{\sigma_p^2} = \frac{\sigma_{ip}}{\sigma_p^2}, \quad (11) \]
• Equation (10) holds for any MVS portfolio $p$ (except the GMVP), i.e., it is not an economic model, and, as such, without economic content.

• It becomes the central equation of the CAPM, however, when we add a few assumptions about investor behavior and market equilibrium.

• To derive (10), first note that

$$
\mu_p - \mu_Z = \mu_p - \frac{b \mu_p - a}{c \mu_p - b} = \frac{c \mu_p^2 - 2b \mu_p + a}{c \mu_p - b} = \frac{d \sigma_p^2}{c \mu_p - b}.
$$

(12)
Also, let \( x_p = (x_{1p}, x_{2p}, \ldots, x_{Np}) \) be the weight vector of \( p \), then, if \( R_i \) is the return of asset \( i \), \( i = 1, \ldots, N \),

\[
\text{Cov}(R_i, R_p) = \text{Cov} \left( R_i, \sum_{j=1}^{N} x_{jp} R_j \right)
\]

\[
= \sum_{j=1}^{N} x_{jp} \text{Cov}(R_i, R_j),
\]

so that the \( i \)th element of \( \sum x_p \) is the covariance between \( R_p \) and \( R_i \).
We obtain from \( x_p = \Sigma^{-1} R \Phi^{-1} \tilde{\mu}_p \)

\[
\Sigma x_p = \Sigma \Sigma^{-1} R \Phi^{-1} \tilde{\mu}_p \\
= (\mu, 1_N)^{\frac{1}{c}} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} \mu_p \\ 1 \end{pmatrix} \\
= (\mu, 1_N) \begin{pmatrix} \frac{c \mu_p - b}{d} \\ \frac{a - b \mu_p}{d} \end{pmatrix} \\
= \frac{c \mu_p - b}{d} \mu - \frac{b \mu_p - a}{d} 1_N \\
= \frac{c \mu_p - b}{d} \left( \mu - \frac{b \mu_p - a}{c \mu_p - b} 1_N \right) \\
= \frac{\sigma^2_p}{\mu_p - \mu_Z} (\mu - \mu_Z 1_N) \quad (13)
\]

since (12)

The \( i \)th equation of (13) is

\[
\text{Cov}(R_i, R_p) = \sigma_{ip} = \frac{\sigma^2_p}{\mu_p - \mu_Z} (\mu_i - \mu_Z). \quad (14)
\]