Problem 1

A risk–averse individual with initial wealth $W_0$ and expected utility function $U$ has two investment alternatives: a risk–free asset with return $r_f$, and a risky asset (or a portfolio of risky assets) with random return $R$. Let $a \geq 0$ be the money amount invested in the risky asset. The wealth in period 1 ($W_1$) is thus

$$W_1 = (1 + r_f)(W_0 - a) + (1 + R)a = (1 + r_f)W_0 + a(R - r_f) = \tilde{W} + a\tilde{R},$$

where $\tilde{W} = (1 + r_f)W_0$, and the excess return over the risk free rate $\tilde{R} = R - r_f$. The investor will maximize expected utility

$$V(a) := E[U(W_1)] = E[U(\tilde{W} + a\tilde{R})].$$

(a) Denote the optimal amount to be invested in the risky asset as $a^\star$. Show that $a^\star > 0 \iff E(R) > r_f$, i.e., there will be positive investment in the risky asset if and only if the expected excess return is positive.

(b) The result in (a) may be interpreted in the sense that in the small (i.e., for small enough payoffs), we (the expected utility maximizers) are all risk neutral. Explain.

(c) Now assume that the investor has a quadratic utility function of the form

$$U(W) = W - \frac{b}{2}W^2, \quad b > 0.$$  

(3)

Solve explicitly for the optimal $a$, i.e., $a^\star$, and show that, for this utility function, $\frac{da^\star}{dW_0} < 0$, i.e., a wealthier individual invests less in risky assets. (Indeed, this is true for all utility functions with increasing absolute risk aversion, $A(W) = -U''(W)/U'(W)$).

(d) Show that $U'''(W) > 0$ is a necessary condition for decreasing absolute risk aversion. (Thus, we can sign the third derivative of reasonable utility functions.)
Problem 2
Consider two agents $A$ and $B$ with quadratic expected utility functions,

$$u_i(w) = w - \frac{b_i}{2}w^2, \quad i = A, B,$$

where $b_A = 0.5$ and $b_B = 0.75$. Both agents have initial wealth $W_0 = 1$. They have to invest this wealth in a portfolio of two risky assets with returns $R_1$ and $R_2$. It is known that $E(R_1) = \mu_1 = 0.25$, $E(R_2) = \mu_2 = 0.05$, $\text{Var}(R_1) = \sigma_1^2 = 1$, and $\text{Var}(R_2) = \sigma_2^2 = 1.5$. The returns are uncorrelated, i.e., $\text{Corr}(R_1, R_2) = \rho_{12} = 0$.

(a) Show that agent $B$ is more risk averse than agent $A$ according to the Arrow–Pratt measure of absolute risk aversion.

(b) Derive the optimal portfolio weight of the first asset for both individuals. Let these be denoted by $x^*_A$ and $x^*_B$ for agents $A$ and $B$, respectively.

(c) The results in b) are $x^*_A = 0.6654$ and $x^*_B = 0.6129$. That is, the more risk averse individual invests less in the asset with the lower variance. Is this in conflict with intuition?

Problem 3
Consider the CARA expected utility function

$$U(W) = -\exp\{-cW\}, \quad c > 0,$$

where $c$ is the constant coefficient of absolute risk aversion. Show that, if wealth is normally distributed with mean $\mu$ and variance $\sigma^2$, expected utility is

$$E[U(W)] = \int_{-\infty}^{\infty} U(W) \phi(W; \mu, \sigma^2) dW = -\exp\left\{-\mu c + \frac{1}{2}c^2\sigma^2\right\},$$

where

$$\phi(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

is the density of a normal variable with mean $\mu$ and variance $\sigma^2$. 

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