II. Black CAPM

Brief Review of the Black CAPM

• The Black CAPM assumes that
  
  (i) all investors act according to the $\mu - \sigma$ rule,
  (ii) face no short selling constraints, and
  (iii) exhibit perfect agreement with respect to the probability distribution of asset returns.

• It is not assumed that they can lend and borrow at a common risk–free rate.

• Under these assumptions, the market portfolio is a mean–variance efficient portfolio.
Thus, there is a portfolio $Z$, i.e., the zero-beta portfolio with respect to the market portfolio, such that for each risky asset or portfolio of risky assets $i$, we have

$$\mu_i = \mu_z + \beta_i (\mu_m - \mu_z),$$

(1)

where $\mu_m$ is the expected return of the market portfolio.
Framework for Estimation and Testing

- The CAPM relationship (1) is expressed in terms of expected values, which are not observable.

- To obtain a model with observable quantities, we describe returns using the following *market model*:

\[
    r_{it} = \alpha_i + \beta_i r_{m,t} + \epsilon_{it} \quad i = 1, \ldots, N
\]  

(2)

\[
    E(\epsilon_{it}) = 0, \quad i = 1, \ldots, N
\]  

(3)

\[
    E(\epsilon_{it}\epsilon_{jt'}) = \begin{cases} 
        \sigma_{ij} & \text{if } t = t' \\
        0 & \text{if } t \neq t'
    \end{cases} \quad i, j = 1, \ldots, N
\]  

(4)

\[
    E(r_{m,t}\epsilon_{i,t}) = 0, \quad i = 1, \ldots, N.
\]  

(5)

- Here \( r_{i,t} \) is the return of asset \( i \) in period \( t \), and \( r_{m,t} \) is the return of the market portfolio in period \( t \).

- This is very similar to the framework employed for testing the Sharpe–Lintner CAPM.
• However, in contrast to the market model considered last week, the relation (2) is *not* stated in terms of excess returns.

• According to equation (4), the asset-specific error terms may be correlated.

• Thus, we allow for a non-diagonal covariance matrix, $\Sigma$, of the vector $\epsilon_t = [\epsilon_{1t}, \ldots, \epsilon_{Nt}]'$,

\[
COV(\epsilon_t) = \Sigma = \begin{bmatrix}
\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\
\sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2
\end{bmatrix}
\]

• Conditional on the excess return of the market, we then also have

\[
COV(r_t) = \Sigma, \quad (6)
\]

where $r_t = [r_{1t}, \ldots, r_{2t}]'$. 

• We will also assume that the error terms follow a multivariate normal distribution, i.e.,

\[
\epsilon_t \overset{iid}{\sim} N(0, \Sigma).
\]  

(7)

• The Black CAPM implies a restriction on the intercept terms in (2), namely,

\[
\begin{align*}
\alpha_i &= (1 - \beta_i)\mu_z, \quad i = 1, \ldots, N, \\
\end{align*}
\]

(8)

or, using vector notation,

\[
\boldsymbol{\alpha} = (\mathbf{1}_N - \boldsymbol{\beta})\mu_z.
\]  

(9)

• Equation (9) imposes a nonlinear restriction on the parameters, because \(\mu_z\) is not known and has to be estimated, along with the further unknown parameters of the (restricted) model, i.e., \(\boldsymbol{\beta}\) and \(\Sigma\).
Estimation of the Parameters

- Write the market model as

$$r_t = \alpha + \beta r_{m,t} + \epsilon_t, \quad t = 1, \ldots, T,$$

$$\epsilon_t \overset{iid}{\sim} N(0, \Sigma),$$

where $$\alpha = [\alpha_1, \ldots, \alpha_N]'$$, and $$\beta = [\beta_1, \ldots, \beta_N]'$$.

- The maximum likelihood estimator (MLE) for the unconstrained model has been derived last week, and is given by

$$\hat{\alpha} = \bar{r} - \hat{\beta}\bar{r}_m,$$  \hspace{1cm} (10)

$$\hat{\beta} = \frac{\sum_{t=1}^{T}(r_t - \bar{r})(r_{m,t} - \bar{r}_m)}{\sum_{t=1}^{T}(r_{m,t} - \bar{r}_m)^2}$$ \hspace{1cm} (11)

$$= \frac{\sum_{t=1}^{T}(r_t - \bar{r})(r_{m,t} - \bar{r}_m)}{T\hat{\sigma}_m^2}$$

$$= \frac{\sum_{t=1}^{T}(r_{m,t} - \bar{r}_m)r_t}{T\hat{\sigma}_m^2},$$
and

\[ \hat{\Sigma} = \frac{1}{T} \sum_{t=1}^{T} \hat{\epsilon}_t \hat{\epsilon}_t' \]  

(12)

\[ = \frac{1}{T} \sum_{t=1}^{T} (r_t - \hat{\alpha} - \hat{\beta}r_{m,t})(r_t - \hat{\alpha} - \hat{\beta}r_{m,t})' \]

where

\[ \bar{r} = \frac{1}{T} \sum_{t=1}^{T} r_t, \quad \bar{r}_m = \frac{1}{T} \sum_{t=1}^{T} r_{m,t}, \]

\[ \hat{\sigma}_m^2 = \frac{1}{T} \sum_{t=1}^{T} (r_{m,t} - \bar{r}_m)^2. \]
Estimation of the Restricted Model

• Recall that the Black CAPM imposes

\[ \alpha = (1_N - \beta) \mu_z. \]  \hfill (13)

• The parameters to estimate are \( \mu_z, \beta \) and \( \Sigma \), and the log–likelihood function is

\[
\log L(\mu_z, \beta, \Sigma) = -\frac{NT}{2} \log(2\pi) - \frac{T}{2} \log |\Sigma| \\
- \frac{1}{2} \sum_{t=1}^{T} \vec{\epsilon}_t \Sigma^{-1} \vec{\epsilon}_t,
\]

where

\[
\vec{\epsilon}_t = r_t - (1_N - \hat{\beta}) \hat{\mu}_z - \hat{\beta} r_{m,t}. \]  \hfill (14)
Note that, by the same arguments as last week, whatever the estimators of $\beta$ and $\mu_z$ will be, the estimator of $\Sigma$ is

$$
\hat{\Sigma}(\hat{\mu}_z, \hat{\beta}) = \frac{1}{T} \sum_{t=1}^{T} \hat{\epsilon}_t \hat{\epsilon}_t' 
$$

(15)

$$
= \frac{1}{T} \sum_{t=1}^{T} (r_t - (1_N - \hat{\beta})\hat{\mu}_z - \hat{\beta} r_{m,t}) 
\times (r_t - (1_N - \hat{\beta})\hat{\mu}_z - \hat{\beta} r_{m,t})'.
$$

Moreover, for any given $\hat{\mu}_z$, $\hat{\beta}$ will be the equation–by–equation OLS estimator of the regression through the origin

$$(r_t - 1_N \hat{\mu}_z) = \beta (r_{m,t} - \hat{\mu}_z), \quad t = 1, \ldots, T.$$
It follows that

$$
\hat{\beta}(\hat{\mu}_z) = \frac{\sum_{t=1}^{T} (r_t - 1_N \hat{\mu}_z)(r_{m,t} - \hat{\mu}_z)}{\sum_{t=1}^{T} (r_{m,t} - \hat{\mu}_z)^2}.
$$

(16)

- From last week’s analysis, we also know that the log-likelihood function, evaluated at the MLE, is

$$
\log L = -\frac{NT}{2} \left[ \log(2\pi) + 1 \right] - \frac{T}{2} \log |\hat{\Sigma}|.
$$

- Thus, we have to find $\hat{\beta}$ and $\hat{\mu}_z$ such that

$$
\log |\hat{\Sigma}| = \log \left| \frac{1}{T} \sum_{t=1}^{T} (r_t - \hat{\mu}_z(1_N - \hat{\beta}) - \hat{\beta} r_{m,t}) \right| \times \left( r_t - \hat{\mu}_z(1_N - \hat{\beta}) - \hat{\beta} r_{m,t} \right)'
$$

(17)

is minimized.
• But, as we have seen in (16), \( \hat{\beta} \) can be written as a function of \( \hat{\mu}_z \), namely

\[
\hat{\beta}(\hat{\mu}_z) = \frac{\sum_{t=1}^{T} (r_t - 1_N \hat{\mu}_z)(r_{m,t} - \hat{\mu}_z)}{\sum_{t=1}^{T} (r_{m,t} - \hat{\mu}_z)^2}.
\]  

(18)

• Thus, (17) can be written as a function of just a single variable, \( \hat{\mu}_z \).

• Hence, we can find the MLE of the restricted model by first identifying \( \hat{\mu}_z \).

• This can be done, for example, by conducting a simple grid–search.

• Then compute \( \hat{\beta} \) via (18) and finally evaluate \( \hat{\Sigma} \) using equation (15).
Likelihood Ratio (LR) Test

- Having estimated the parameters of both the unrestricted as well as those of the restricted market model, we can conduct a likelihood ratio (LR) test.

- If
  - \( \hat{\Sigma}_1 \) denotes the estimated error term covariance matrix under the unrestricted model, and
  - \( \hat{\Sigma}_0 \) is the estimated error term covariance matrix under the null hypothesis,

then, by the same arguments as last week, the LR test statistic is

\[
\mathcal{LR} = T \left[ \log |\hat{\Sigma}_0| - \log |\hat{\Sigma}_1| \right] \overset{as}{\sim} \chi^2(N - 1).
\]

- Note that the degrees of freedom of the null distribution is \( N - 1 \). Relative to the Sharpe–Lintner model, we lose one degree of freedom because \( \mu_z \) is a free parameter.
• We have also seen that the asymptotic likelihood ratio test may exhibit poor performance in finite samples.
• To mitigate these effects, the adjusted statistic

\[ \mathcal{L}\mathcal{R}^* = \left( T - \frac{N}{2} - 2 \right) \left[ \log |\hat{\Sigma}_0| - \log |\hat{\Sigma}_1| \right] \]

\[ \text{asy} \sim \chi^2(N - 1) \]

has been shown to more closely match the \( \chi^2 \) distribution in finite samples.

• There also exists a further device that provides a useful check.\(^1\)

• This also gives rise to a closed–from estimator for the zero-beta rate, \( \mu_Z \).

Lower Bound for the Exact Distribution

• Suppose for the moment that $\mu_z$ is known.

• Then, we can proceed as last week when testing the Sharpe–Lintner model, i.e., we can consider the “excess return” market model

$$r_t - \mu_z 1_N = \alpha + \beta(r_m,t - \mu_z) + \epsilon_t. \quad (19)$$

• The zero–beta CAPM is true if $\alpha = 0$.

• The estimates of the unrestricted model are

$$\hat{\alpha}_1 = \bar{r} - 1_N \mu_z - \hat{\beta}(\bar{r}_m - \mu_z)$$

$$\hat{\beta}_1 = \frac{\sum_t (r_t - \bar{r})(r_{m,t} - \bar{r}_m,t)}{\sum_t (r_{m,t} - \bar{r}_m)^2}$$

$$\hat{\Sigma}_1 = \frac{1}{T} \sum_t [r_t - \bar{r} - \hat{\beta}_1 (r_{m,t} - \bar{r}_m)]$$

$$\times [r_t - \bar{r} - \hat{\beta}_1 (r_{m,t} - \bar{r}_m)]'$$.
Note that $\hat{\beta}_1$ and $\hat{\Sigma}_1$ do not depend on $\mu_z$, and, thus, the value of the log–likelihood function does also not depend on $\mu_z$, as, at the maximum,

$$\log L_1 = -\frac{N T}{2} [\log(2\pi) + 1] - \frac{T}{2} \log |\hat{\Sigma}_1|.$$ 

- The MLE under the restriction that $\alpha = 0$ is

$$\hat{\beta}_0(\mu_z) = \frac{\sum_t (r_t - 1_N \mu_z)(r_{m,t} - \mu_z)}{\sum_t (r_{m,t} - \mu_z)^2} \quad (20)$$

$$\hat{\Sigma}_0(\mu_z) = \frac{1}{T} \sum_t (r_t - \mu_z(1_N - \hat{\beta}_0) - \hat{\beta}_0 r_{m,t}) \times (r_t - \mu_z(1_N - \hat{\beta}_0) - \hat{\beta}_0 r_{m,t})'.$$
The value of the constrained log–likelihood is

\[
\log L_0(\mu_z) = -\frac{NT}{2} \left[ \log(2\pi) + 1 \right] - \frac{T}{2} \log |\Sigma_0(\mu_z)|,
\]

which can be viewed as a function of only one variable, i.e., \( \mu_z \).

Consequently, the likelihood ratio test statistic,

\[
\mathcal{LR}(\mu_z) = T \left[ \log |\hat{\Sigma}_0(\mu_z)| - \log |\hat{\Sigma}_1| \right],
\]  \hspace{1cm} (21)

can be viewed as a function of only \( \mu_z \).

Obviously, the value of \( \mu_z \) which minimizes the likelihood ratio statistic will be the MLE of \( \mu_z \).

(Recall that \( |\hat{\Sigma}_1| \) does not depend on \( \mu_Z \).)
• In the last week, we have developed a relation between the LR test and the $F$ test for the Sharpe–Linter CAPM, which used a formula expressing $|\Sigma_0|$ in terms of $|\Sigma_1|$ and $\hat{\alpha}$.

• Repeating the same line of arguments shows that (21) can be written as

$$\mathcal{LR}(\mu_z) = T \log \left[ \hat{\alpha}' \hat{\Sigma}_1^{-1} \hat{\alpha} \frac{\hat{\sigma}_m^2}{(\bar{r}_m - \mu_z)^2 + \hat{\sigma}_m^2} + 1 \right] ,$$

where

$$\hat{\alpha} = (\bar{r} - \hat{\beta}_1 \bar{r}_m) - (1_N - \hat{\beta}_1) \mu_z. \quad (23)$$

is a function of $\mu_z$
• Thus, the MLE of $\mu_z$ is the value which minimizes

\[
g(\mu_z) = \hat{\alpha}'\hat{\Sigma}_1^{-1}\hat{\alpha} \frac{\hat{\sigma}_m^2}{(\bar{r}_m - \mu_z)^2 + \hat{\sigma}_m^2}
\]

\[
= \frac{[\mu_z^2 a - 2b\mu_z + c]\hat{\sigma}_m^2}{\hat{\sigma}_m^2 + (\bar{r}_m - \mu_z)^2},
\]

where

\[
a = (1_N - \hat{\beta}_1)'\hat{\Sigma}_1^{-1}(1_N - \hat{\beta}_1),
\]

\[
b = (1_N - \hat{\beta}_1)'\hat{\Sigma}_1^{-1}(\bar{r} - \hat{\beta}_1\bar{r}_m),
\]

\[
c = (\bar{r} - \hat{\beta}_1\bar{r}_m)'\hat{\Sigma}_1^{-1}(\bar{r} - \hat{\beta}_1\bar{r}_m).
\]
It follows that we can find the MLE of \( \mu_z \) by solving

\[
\frac{dg(\mu_z)}{d\mu_z} = \frac{(2a\mu_z - 2b)[(\bar{r}_m - \mu_z)^2 + \hat{\sigma}_m^2]}{[(\bar{r}_m - \mu_z)^2 + \hat{\sigma}_m^2]^2} - \frac{(\mu_z^2 a - 2b\mu_z + c)2(\mu_z - \bar{r}_m)}{[(\bar{r}_m - \mu_z)^2 + \hat{\sigma}_m^2]^2} = 0,
\]

that is, \( \mu_z \) is a root of the quadratic

\[
A\mu_z^2 + B\mu_z + C = 0, \tag{24}
\]

where

\[
A = b - ar_m,
B = a(\hat{\sigma}_m^2 + \bar{r}_m^2) - c,
C = -b(\hat{\sigma}_m^2 + \bar{r}_m^2) + cr_m.
\]

This is a closed–form solution for \( \hat{\mu}_z \).
• It can be shown that the roots of (24) are real.

• The maximum likelihood estimator, then, is the root which corresponds to the smaller value of \( \log(\det(\hat{\Sigma}(\mu_z))) \).

• Once \( \hat{\mu}_z \) is determined, we can apply the closed form expressions (20) to determine the MLE for the other parameters.
• Now let us return to our objective of finding a test which does not rely on asymptotic arguments.

• Recall that, with $\mu_z$ known, we could use the statistic

\[
J(\mu_z) = \frac{T - N - 1}{N} \left[ 1 + \frac{(\bar{r}_m - \mu_z)^2}{\hat{\sigma}_m^2} \right]^{-1} \hat{\alpha}' \hat{\Sigma}_1^{-1} \hat{\alpha}
\]

(25)

to conduct an exact finite–sample test.

• This uses the result that, in this case, (25) has an $F$ distribution with $N$ degrees of freedom in the numerator and $T - N - 1$ degrees of freedom in the denominator.

• As $\mu_z$ is not known, this cannot be done.

• The MLE of $\mu_z$, $\hat{\mu}_z$, is the value which minimizes the LR test statistic, $\mathcal{LR}(\mu_z)$.

• As we know that $J(\mu_z)$ is a monotonic transformation of $\mathcal{LR}(\mu_z)$, $\hat{\mu}_z$ also minimizes $J(\mu_z)$. 
Thus,

\[ J(\hat{\mu}_z) \leq J(\mu_z), \quad (26) \]

where \( \mu_z \) is the true value of the zero–beta portfolio’s mean.

That is, the \( F \) test based on \( \hat{\mu}_z \) and using the “exact” \( F \) distribution will accept too often.

But we know that, if it rejects, it will reject for any value of \( \mu_z \), and we need not resort to asymptotic approximations in this case.

This is a useful check because we have seen that the asymptotic likelihood ratio test rejects too often.