Exercises III

1. Given the p.m.f of random variable X below, find the distribution of $Y = X^2$.

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(x)</td>
<td>1/5</td>
<td>1/6</td>
<td>1/5</td>
<td>1/15</td>
<td>11/30</td>
</tr>
</tbody>
</table>

2. Let random variable X have Uniform distribution with parameters (0,1) and $Y = e^X$. Find the density of Y. Also, given $Y = -\ln X$, determine the density of Y.

3. Let random variable X have standard normal distribution and $Y = X^2$. Find the density of Y.

4. Let $X_i \sim N(\mu, \sigma^2)$ be a sequence of Normal rvs. Let $\bar{X}_n$ be the sample mean. Find the limiting distribution of $\bar{X}_n$.

5. Let $X_i \sim \text{Gamma}(n, \beta)$ where $\beta$ does not depend on $n$. Let $Y_n = X_i/n$. Find the limiting distribution of $Y_n$.

6. Let $X_i \sim \text{Exp}(1)$ and $\bar{X}_n$ be the sample mean of r.s. of size $n$. Find the limiting distribution of $\sqrt{n}((\bar{X}_n - 1))$.

7. Let $\bar{X}_n$ be the sample mean from a r.s. of size $n=100$ from $X^2$ with d.f. 50. Compute approximate value of $P(\bar{X}_n << 51)$.

8. $X \sim \text{Uniform}(0, \theta)$. A r.s. of size $n$ is taken. $X_{(n)}$ is the largest order statistic. Then,
   a) Find the limiting distribution of $X_{(n)}$.
   b) b) Find the limiting distribution of $Z_n = n(\theta - X_{(n)})$.

9. $X \sim \text{Exp}(\theta)$. For a r.s of size $n$, find the MLE of $\theta$.

10. $X \sim \text{Gamma}(\alpha, \beta)$. For a r.s of size $n$, find the MMEs of $\alpha$ and $\beta$.

11. $X \sim \text{Exp}(\theta)$, $\theta \sim 0$. For a r.s of size $n$, find the MLE of $\theta$.

12. $X \sim N(\mu, \sigma^2)$. For a r.s of size $n$, find the MLEs of $\mu$ and $\sigma^2$.

13. $X \sim \text{Uniform}(0, \theta)$, $\theta \sim 0$. For a r.s of size $n$, find the MLE of $\theta$.

14. $X \sim N(\mu, \sigma^2)$. For a r.s of size $n$, the MLE of $\mu$ is $\bar{X}$. Find the MLE of $\mu^2$.

15. $X \sim \text{Uniform}(0, \theta)$, $\theta \sim 0$. For a r.s of size $n$, is the MLE of $\theta$ an Unbiased Estimator and a Consistent Estimator of $\theta$?

16. Let $X_1, X_2, ..., X_n$ be a r.s. from $NB(2, \theta)$ distribution. a) Find the MLE of $\theta$. b) MLE of $\mu = E[X]$.

17. Let $X_1, X_2, ..., X_n$ be a r.s. from $\text{Uniform}(0, \theta)$ distribution. a) Find the MLE of $\theta$, b) Find the MME of $\theta$, c) Compare MSEs of MLE and MME of $\theta$ and comment on which one is better estimator of $\theta$.

18. $X \sim N(\mu, \sigma^2)$ where $\mu$ is known. For a r.s. of size $n$, find a sufficient statistics for $\sigma^2$.

19. Let $X_1, X_2, ..., X_n$ be a r.s. from Cauchy($\theta$) distribution. Find a sufficient statistics for $\theta$, if exists.

20. $X \sim N(\mu, \sigma^2)$. For a r.s of size $n$, find jointly sufficient statistics for $\mu$ and $\sigma^2$.

21. $X \sim \text{Poisson}(\theta)$. Find a consistent sufficient statistics for $\theta$ and find the Minimum Variance Unbiased Estimator of $P(X \leq 1) = 1 + \theta e^{-\theta}$.

22. Given $X \sim \text{Poi}(\mu)$. A. Find CRLB for $\mu$. B. Find Cramer Rao Lower Bound (CRLB) for $e^\mu$.

23. $X \sim \text{Exp}(1/\theta)$. Find an Efficient Estimator of $\theta$, if exists.

24. $X \sim \text{Exp}(\theta)$. Assume that we have a r.s of size $n$, a. Find a Fisher Information. B. Using CRLB, show that $\bar{X}$ is the Efficient Estimator of $\theta$.
25. Let $X_1, X_2, \ldots, X_n$ be a r.s. of $\text{Exp}(\theta)$, $\theta > 0$. Find a $100\gamma\%$ Confidence Interval for $\theta$. Interpret the result.

26. Let $X_1, X_2, \ldots, X_n$ be a r.s. of $N(\mu, \sigma^2)$. Find a $100\gamma\%$ CI for $\mu$ and $\sigma^2$. Interpret the results.

27. To estimate the amount of lumber that can be harvested in a tract of land, the mean diameter of trees in the tract must be estimated to within one inch with 99% confidence. What sample size should be taken? Assume that diameters are normally distributed with $s = 6$ inches.

28. A new breakfast cereal is test-marked for 1 month at stores of a large supermarket chain. The result for a sample of 16 stores indicate average sales of $1200$ with a sample standard deviation of $180$. Set up 99% confidence interval estimate of the true average sales of this new breakfast cereal. Assume normality.

29. An investor is trying to estimate the return on investment in companies that won quality awards last year. A random sample of 83 such companies is selected, and the return on investment is calculated had he invested in them. Construct a 95% confidence interval for the mean return.

30. A random sample of 22 observations from a normal population possessed a variance equal to 37.3. Find 90% CI for $\sigma^2$.

31. The weights of pots of jam made by a standard process is normally distributed with mean $\mu = 345$gr and $\sigma = 2.8$gr. A pot produced just before the process closed for the day weight 338.5gr. Is the process working correctly? $\alpha = 0.01$

32. Do the contents of bottles of catsup have a net weight below an advertised threshold of 16 ounces? To test this 25 bottles of catsup were selected. They gave a net sample mean weight of 15.9. It is known that the standard deviation is 0.4. Test this hypothesis at significance levels 1% and 5%.

33. Mom’s Home Cokin’ claims that 70% of the customers are able to dine for less than $5. Mom wishes to test this claim at the 92% level of confidence. A random sample of 110 patrons revealed that 66 paid less than $5 for lunch. Find if the claim is correct.

34. Proctor and Gamble told its customers that the variance in the weights of its bottles of Pepto-Bismol is less than 1.2 ounces squared. As a marketing representative for P&G, you select 25 bottles and find a variance of 1.7. At the 10% level of significance, is P&G maintaining its pledge of product consistency?

35. A manufacturer claims that compared with his closest competitor, fewer of his employees are union members. Of 318 of his employees, 117 are unionists. From a sample of 255 of the competitor’s labor force, 109 are union members. Perform a test at $\alpha = 0.05$.

36. In a study, doctors discovered that aspirin seems to help prevent heart attacks. Half of 22,000 male physicians took aspirin and the other half took a placebo. After 3 years, 104 of the aspirin and 189 of the placebo group had heart attacks. Test whether this result is significant.

37. $X \sim N(\mu, \sigma^2)$ where $\sigma^2$ is known. Given $H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1$, find the most powerful test of size $\alpha$.

38. $X \sim \text{Uniform}(0, \theta)$. Given a. Single observation, b) Random sample of size $n$ and $H_0: \theta = 3$ vs. $H_1: \theta > 3$, find UMPT of size $\alpha$.

39. Given $X \sim \text{Exp}(\theta)$, $H_0: \theta \leq \theta_0$ vs. $H_1: \theta > \theta_0$, find UMPT of size $\alpha$. 