Chapter 12
HYPOTHESIS TESTING

A statistical hypothesis is a statement about the distribution of random variable $X$. If the hypothesis completely specifies the distribution, we can write the test as

$$H_0 : \theta = \theta_0 \Rightarrow \text{simple hypothesis} \quad \text{which is equivalent to saying} \quad H_0 : f(x;\theta_0) = f(x;\theta_0)$$

$$H_1 : \theta = \theta_1$$

$$H_0 : \theta \geq \theta_0 \quad H_0 : \theta \leq \theta_0 \quad H_0 : \theta = \theta_0$$

$$H_1 : \theta < \theta_0 \quad H_1 : \theta > \theta_0 \quad H_1 : \theta \neq \theta_0$$

$$H_0 : \theta \geq \theta_0 \quad H_0 : \theta \leq \theta_0$$

$$H_1 : \theta < \theta_0 \quad H_1 : \theta > \theta_0$$

$$H_0 : \theta = \theta_0$$

$$H_1 : \theta \neq \theta_0$$

$$\{\text{composite hypotheses}\}$$

$$\{\text{one tailed tests}\}$$

$$\{\text{two tailed test}\}$$

Critical region is the subset of sample space that corresponds to rejecting the null hypothesis.

**Type I error** is rejecting $H_0$ when it is true. **Type II error** is failing to reject a false $H_0$.

$$\Pr(\text{Type I error}) = \alpha \quad \text{and} \quad \Pr(\text{Type II error}) = \beta$$

For simple $H_0$, the probability of rejecting a true $H_0$ is referred to as the **significance level**.

For composite $H_0$, the size of the test (critical region) is the maximum probability of rejecting $H_0$ when it is true.

**Standard approach** is first to select some acceptable level of $\alpha$, and determine the value of critical value. Among all critical regions of size $\alpha$, we choose the one with smallest $\beta$. The **power function** $\pi(\theta)$ is the probability of rejecting $H_0$ when the true value of the parameter is $\theta$.

Corresponding to an observed value of a test statistic, the **p-value** is the lowest level of significance at which the null hypothesis could have been rejected.
Exercises:

1. Let \( X \) have a p.d.f of the form \( f(x; \theta) = \theta x^{\theta-1}, \ 0 < x < 1, \) zero elsewhere, where \( \theta \in \{ \theta : \theta = 1, 2 \} \). To test the simple hypothesis \( H_0 : \theta = 1 \) against \( H_1 : \theta = 2 \), use a random sample of size \( n=2 \) and define the critical region to be \( C = \{(x_1, x_2) : \frac{3}{4} \leq x_1x_2 \} \). Find the power function of the test.

2. Assume that the weight of a cereal in a '10-ounce box' is \( N(\mu, \sigma^2) \). To test \( H_0 : \theta = 10.1 \) against \( H_1 : \theta > 10.1 \), we take a random sample of size \( n=16 \) and observe that \( \bar{x} = 10.4 \) and \( s = 0.4 \).
   
   a. Do we accept or reject \( H_0 : \theta = 10.1 \) at the 5% significance level?
   
   b. What is the approximate p-value of this test?

Neyman-Pearson Theorem:

A critical region for testing a simple null hypothesis \( \theta = \theta_0 \) against a simple alternative hypothesis \( \theta = \theta_1 \) is said to be best or most powerful, if the power of the test at \( \theta = \theta_1 \) is a maximum (Freund).

Let \( X_1, X_2, \ldots, X_n \) be a random sample having p.d.f. \( f(x; \theta) \). Then the joint p.d.f. of \( X_1, X_2, \ldots, X_n \) is \( L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \theta) \)

Lemma: Let \( \theta_1, \theta_2 \) be distinct fixed values of \( \theta \) so that \( \Omega = \{ \theta : \theta = \theta_1, \theta_2 \} \) and let \( k \) be the positive number. Let \( C \) be a subset of the sample space stating a critical region of size \( \alpha \). If

a. \( \frac{L(\theta_0; x_1, x_2, \ldots, x_n)}{L(\theta_1; x_1, x_2, \ldots, x_n)} \leq k, \) for each point \( (x_1, x_2, \ldots, x_n) \in C \).

b. \( \frac{L(\theta_0; x_1, x_2, \ldots, x_n)}{L(\theta_1; x_1, x_2, \ldots, x_n)} \geq k, \) for each point \( (x_1, x_2, \ldots, x_n) \in C^c \), where \( C^c \) denotes the complement of the region.

then \( C \) is a most powerful (MP) test of size \( \alpha \) for testing \( \theta = \theta_0 \) against \( \theta = \theta_1 \).
Exercises:
3. A single observation of a random variable having a geometric distribution is to be used to test the null hypothesis that its parameter equals $\theta_0$ against $\theta > \theta_0$. Use Neyman Pearson lemma to find a best critical region of size $\alpha$.

4. Given a random sample of size $n$ from a normal population with $\mu = 0$ use the Neyman Pearson lemma to construct the most powerful critical region of size $\alpha$ to test the null hypothesis $\sigma = \sigma_0$ against $\sigma = \sigma_1 > \sigma_0$.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample having p.d.f of the form $f(x; \theta) = \theta x^{\theta-1}, \ 0 < x < 1$, zero elsewhere. Show that the best critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$, is $C = \{(x_1, x_2) : c \leq \prod_{i=1}^{n} x_i\}$.

Definition: The critical region $C$ is a Uniformly Most Powerful critical region of size $\alpha$ for testing simple hypothesis $H_0$ against a composite alternative $H_1$ if the set $C$ is a best critical region of size $\alpha$ for testing simple hypothesis $H_0$ against a simple alternative $H_1$.

The Power Function of a Test
The power function of a test of a statistical hypothesis $H_0$ against $H_1$ is given by
\[ \pi(\theta) = \begin{cases} \alpha(\theta) & \text{under } H_0 \\ 1 - \beta(\theta) & \text{under } H_1 \end{cases} \]

Likelihood Ratio Tests
A method of constructing a test of composite hypothesis against an alternative composite hypothesis or of constructing a test of simple hypothesis against an alternative composite hypothesis when a uniformly most powerful test does not exist (Hogg and Craig)

Definition: Suppose $X_1, X_2, \ldots, X_n$ constitute a random sample of size $n$ from a population having density function $f(x; \theta)$. Let $\omega$ denote the parameter space for $\theta$. We test $H_0 : \theta \in \omega$ against $H_1 : \theta \in \Omega$. Define the likelihood functions
\[ L(\theta; x_1, x_2, \ldots, x_n), \ \theta \in \omega \text{ and } L(\theta; x_1, x_2, \ldots, x_n), \ \theta \in \Omega. \]
Let \( L(\hat{\omega}; x_1, x_2, \ldots, x_n) \) and \( L(\hat{\Omega}; x_1, x_2, \ldots, x_n) \) be the maxima, which we assume to exist, of these two likelihood functions. The ratio
\[
\frac{L(\hat{\omega}; x_1, x_2, \ldots, x_n)}{L(\hat{\Omega}; x_1, x_2, \ldots, x_n)} = \lambda
\]
is called the \textbf{Likelihood ratio}.

Given \( k; \ 0 < k < 1, \ \lambda \leq k \) defines a \textbf{likelihood ratio test} of size \( \alpha \).

**Theorem:** For large \( n \), the distribution of \(-2 \ln \lambda \) is \( \chi^2 \) with 1 degrees of freedom.

**Exercises:**
6. A random sample of size \( n \) is to be used to test the null hypothesis that the parameter \( \theta \) of an exponential population equals to \( \theta_0 \) against the alternative that it does not equal to \( \theta_0 \).
   
   a. Find an expression for the likelihood ratio statistic.
   
   b. Use the result of part (a) to show that the critical region of the likelihood ratio test can be written as \( \bar{x}e^{-\theta_0} \leq K \).

7. A random sample \( X_1, X_2, \ldots, X_n \) arises from a distribution given by
\[
H_0: f(x; \theta) = \frac{1}{\theta}, \quad 0 < x < \theta, \quad \text{zero elsewhere.}
\]
\[
H_1: f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty, \quad \text{zero elsewhere.}
\]

Determine the likelihood ratio test associated with the test of \( H_0 \) against \( H_1 \).