Chapter 3
Probability Distribution and Density

Random Variable
Let $\Omega$ be human population containing $n$ individuals
Age distribution: $A(w) \rightarrow \text{age of } w$
$w \rightarrow A(w)$
Height distribution: $H(w) \rightarrow \text{height of } w$
$w \rightarrow H(w)$
We may consider combination of age and height (mixed distribution)
$w \rightarrow A(w) + \mu H(w)$
$\lambda, \mu \text{ constant}$
Let the concerned age interval be $18 \leq A(w) \leq 40$, height interval be $145 \leq H(w) \leq 199$, then the joint sample space will be
\[
\{w \mid 18 \leq A(w) \leq 40\} \cap \{w \mid 145 \leq H(w) \leq 199\}
\]

Definition A numerically valued function $X$ of $w$ with domain $\Omega$
$w \in \Omega : w \rightarrow X(w)$ is called a random variable (r.v).

Proposition If $X$ and $Y$ are random variables, then
$X + Y; \ X - Y; \ XY; \ \frac{X}{Y}$ $(Y \neq 0)$ and $aX + bY$ are random variables.

Distribution Function
Definition: Let $X$ be a discrete random variable and $x_1, \ldots, x_n$ be possible observations from $X$, the probability distribution is $p_n = \Pr(X = x_n)$.
For any values $a$ and $b$,
\[
\Pr(a \leq X \leq b) = \sum_{a \leq x \leq b} p_n
\]
Assume an infinite interval $(-\infty, x]$ then
\[
\Pr(X \leq x) = F_X(x) = \sum_{x \leq x} p_n
\]
$X \rightarrow F_X(X)$ defined on $\mathbb{R}^1$ is called distribution function of $X$
It picks up all the probabilities of values of $X$ up to $x$ (inclusive)
\[
P(a \leq X \leq b) = F_X(b) - F_X(a)
\]
Properties: i) $\forall n: P_n \geq 0$

   ii) $\sum P_n = 1$

   iii) $\sum P(X = x_i) = P(\Omega) = 1$

Definition: Distribution function (d.f) $F(x)$ is defined as

- $F(X) = P(X \leq x) = \sum f(w)$ for discrete $X$

- $F(X) = P(X \leq x) = \int_{-\infty}^{x} f(w)dw$ for continuous $X$

Properties
1. $0 \leq F(X) \leq 1$
2. $F(X)$ is non-decreasing function of $x$
3. $F(\infty) = 1$ and $F(-\infty) = 0$
4. $F(X)$ is continuous to the right at each point $x$.

If $x$ is a discontinuity point of $F(x)$, $P(X = x) = \text{jump}$ which the distribution function has at point $x$. If $x$ is a continuity point of $F(x)$, $P(X = x) = 0$

If $A = (-\infty, x]$

$F(X) = P(X \leq x) = \int_{-\infty}^{x} f(w)dw$ distribution function

If $X$ is continuous and $F(x)$ is differentiable then

$F'(x)\frac{\partial F(x)}{\partial x} = f(x)$

Example: Freund (5th ed) 3.2.b

$f(x) = \frac{x^2}{30}$ for $x = 0, 1, 2, 3, 4$

$p_0 = \Pr(X = 0) = f(0) = 0$ \hspace{1cm} $p_1 = \Pr(X = 1) = f(1) = \frac{1}{30}$

$p_2 = \Pr(X = 2) = f(2) = \frac{4}{30}$ \hspace{1cm} $p_3 = \Pr(X = 3) = f(3) = \frac{9}{30}$

$p_4 = \Pr(X = 4) = f(4) = \frac{16}{30}$

Question: Is $f(x)$ a p.d.f?

$\sum_{i=0}^{4} p_i = \sum_{i=0}^{4} f(i) = \frac{1+4+9+16}{30} = 1 \hspace{1cm} \text{yes!}$

Question: Find the Cumulative Probability Distribution
\( F(x) = \Pr(X \leq x) \)

for \( x = 0 \)  \( F(0) = \Pr(X \leq 0) = 0 \)

for \( x = 1 \)  \( F(1) = \Pr(X \leq 1) = f(0) + f(1) = \frac{1}{30} \)

for \( x = 2 \)  \( F(2) = \Pr(X \leq 2) = f(0) + f(1) + f(2) = \frac{1}{30} + \frac{4}{30} = \frac{5}{30} \)

for \( x = 3 \)  \( F(3) = \Pr(X \leq 3) = f(0) + f(1) + f(2) + f(3) = \frac{1}{30} + \frac{4}{30} + \frac{9}{30} = \frac{14}{30} \)

for \( x = 4 \)  \( F(4) = \Pr(X \leq 4) = f(0) + f(1) + f(2) + f(3) + f(4) = \frac{1}{30} + \frac{4}{30} + \frac{9}{30} + \frac{16}{30} = 1 \)

\[
F(x) = \begin{cases} 
0 & 0 \leq x < 1 \\
\frac{1}{30} & 1 \leq x < 2 \\
\frac{5}{30} & 2 \leq x < 3 \\
\frac{14}{30} & 3 \leq x < 4 \\
\frac{1}{30} & x \geq 4 
\end{cases}
\]

Example: Freund (5th ed) 3.35

\[
\begin{cases} 
\frac{x^2}{30} & 0 < x \leq 1 \\
\frac{1}{2} & 1 < x \leq 2 \\
\frac{3-x}{2} & 2 < x \leq 3 
\end{cases}
\]

Given \( f(x) \) find cumulative distribution function of the piecewise function.
\[ F(x) = \Pr(X \leq x) \]

for \( x = 0 \quad F(0) = \Pr(X \leq 0) = 0 \)

for \( 0 \leq x < 1 \quad F(x) = \Pr(X \leq x) = \int_0^x f(w)dw + \int_0^1 \frac{1}{2} dw = \frac{x^2}{4} \]

for \( 1 \leq x < 2 \quad F(x) = \Pr(X \leq x) = \int_0^1 \frac{w}{2} dw + \int_0^1 \frac{1}{2} dw = \frac{x-1}{2} \]

for \( 2 \leq x < 3 \quad F(x) = \Pr(X \leq x) = \int_0^2 \frac{w}{2} dw + \int_2^3 \frac{1}{2} dw + \int_2^3 \frac{3-w}{2} dw = \frac{3}{4} + \left( \frac{6x-x^3-8}{4} \right) \]

for \( x \geq 3 \quad F(x) = \Pr(X \leq x) = \int_0^2 \frac{w}{2} dw + \int_2^3 \frac{1}{2} dw + \int_2^3 \frac{3-w}{2} dw = 1 \]

**Multivariate Distributions**

**Definition:**
Given \( X \) and \( Y \) are discrete random variables, the **joint probability distribution** of \( X \) and \( Y \) is
\[ f(x, y) = \Pr(X = x, Y = y) \]

**Theorem:** \( f(x, y) \) satisfies the following conditions:
1. \( f(x, y) \geq 0 \) for all values of \( X \).
2. \( \sum_{x, y} f(x, y) = 1 \)

**Definition:**
Given \( X \) and \( Y \) are discrete random variables, the **joint distribution function** or **joint cumulative distribution** of \( X \) and \( Y \) is
\[ F(x, y) = \Pr(X \leq x, Y \leq y) = \sum_{s \leq x} \sum_{t \leq y} f(s, t) \]

**Definition:**
Given \( X \) and \( Y \) are continuous random variables, the **joint probability density** of \( X \) and \( Y \) is
\[ \Pr((X, Y) \in A) = \iint_A f(x, y)dxdy \quad \text{for any region in XY-plane} \]

**Theorem:** \( f(x, y) \) satisfies the following conditions:
1. \( f(x, y) \geq 0 \) for all values of \( X \).
2. \( \int \int \int f(x, y)dxdy = 1 \).
Definition:
Given X and Y are continuous random variables, the joint distribution function or joint cumulative distribution of X and Y is
\[ F(x, y) = \Pr(X \leq x, Y \leq y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s, t) ds dt. \]

Marginal Distributions

Definition: If X and Y are random variables having joint distribution (density) of \( f(x,y) \)
\[ f(x) = \int_{-\infty}^{\infty} f(x,y) dy \] is the marginal distribution of \( x \).
\[ f(y) = \int_{-\infty}^{\infty} f(x,y) dx \] is the marginal distribution of \( y \).

Conditional Distribution

Conditional density of \( X \) given \( Y=y \) is given by
\[ f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}; \quad f(y) \neq 0 \]

Theorem: Let X and Y be two random variables having joint p.d.f. \( f(x,y) \). X and Y are said to be independent iff \( f(x,y) = f(x)f(y) \)

Theorem: Let \( X_1, X_2, \ldots, X_n \) be two random variables having joint p.d.f. \( f(x_1, x_2, \ldots, x_n) \). \( X_1, X_2, \ldots, X_n \) are said to be independent iff \( f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n) = \prod_{i=1}^{n} f(x_i) \)

Theorem: Let X and Y be two random variables having joint p.d.f. \( f(x,y) \). X and Y are said to be independent iff \( f(x|y) = f(x) \) and \( f(y|x) = f(y) \)

Example: Freund (5th ed) 3.54
\[ f(x,y) = c(x^2 + y^2) \quad x = -1, 0, 3 ; \ y = -1, 2, 3 \]

Question: If \( f(x,y) \) is a p.d.f, find the value of \( c \).
\[ \sum_{x=-1}^{3} \sum_{y=-1}^{3} f(x, y) = f(-1,-1) + f(-1,2) + f(-1,3) + f(0,-1) + f(0,2) + f(0,3) + f(1,-1) + f(1,2) + f(1,3) + f(3,-1) + f(3,2) + f(3,3) = 1 \]
\[ \sum_{x=-1}^{3} \sum_{y=-1}^{3} f(x, y) = 89c = 1 \Rightarrow c = \frac{1}{89} \Rightarrow f(x,y) = \frac{x^2 + y^2}{89} \quad x = -1, 0, 3 ; \ y = -1, 2, 3 \]

Question: Find the marginal probability of X:
\[ f(x) = \sum_{y=1}^{3} f(x, y) = f(x, -1) + f(x, 2) + f(x, 3) \]

\[
x = -1 \quad f(-1) = f(-1, -1) + f(-1, 2) + f(-1, 3) = \frac{17}{89}
\]

\[
x = 0 \quad f(0) = f(0, -1) + f(0, 2) + f(0, 3) = \frac{14}{89}
\]

\[
x = 1 \quad f(1) = f(1, -1) + f(1, 2) + f(1, 3) = \frac{17}{89}
\]

\[
x = 3 \quad f(3) = f(3, -1) + f(3, 2) + f(3, 3) = \frac{41}{89}
\]

**Question:** Find the marginal probability of Y:

\[ f(y) = \sum_{x=-1}^{3} f(x, y) = f(-1, y) + f(0, y) + f(1, y) + f(3, y) \]

\[
y = -1 \quad f(-1) = f(-1, -1) + f(0, -1) + f(1, -1) + f(3, -1) = \frac{15}{89}
\]

\[
y = 2 \quad f(2) = f(-1, 2) + f(0, 2) + f(1, 2) + f(3, 2) = \frac{27}{89}
\]

\[
y = 3 \quad f(3) = f(-1, 3) + f(0, 3) + f(1, 3) + f(3, 3) = \frac{47}{89}
\]

**Question:** Find the following probabilities:

\[
\Pr(X=1,Y=2) = f(1,2) = \frac{5}{89}
\]

\[
\Pr(X = 0, 1 \leq Y < 3) = f(0, 2) = \frac{4}{89}
\]

\[
\Pr(X + Y \leq 1) = f(-1, -1) + f(-1, 2) + f(0, -1) + f(1, -1) = \frac{10}{89}
\]

\[
\Pr(X > Y) = f(0, -1) + f(1, -1) + f(3, 2) = \frac{16}{89}
\]

**Question:** Find conditional distribution of X given Y

\[
f(x | y) = \frac{f(x, y)}{f(y)}
\]

\[
y = -1 \Rightarrow f(x | y = 1) = \frac{f(x, 1)}{f(1)}
\]

\[
\Rightarrow \text{for } x = -1 \Rightarrow f(-1 | -1) = \frac{2/89}{15/89} = \frac{2}{15}
\]

\[
\Rightarrow \text{for } x = 0 \Rightarrow f(0 | -1) = \frac{5/89}{15/89} = \frac{5}{15}
\]

and so on.
**Question:** Are X and Y independent?
Check if $f(-1,-1)=f(-1)f(-1)$?

$$f(-1,-1) = \frac{2}{89}$$

$$f_x(-1) = \frac{17}{89}$$

$$f_y(-1) = \frac{15}{89}$$

$$\frac{2}{89} \neq \frac{17}{89} \times \frac{15}{89}$$

**NO!**

Example: Freund (5th ed) 3.66

$$F(x, y) = 1 - (e^{-x} + e^{-y}) + e^{-x-y} \quad x > 0; \ y > 0.$$  

**Question:** Find joint density function.

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} (1 - (e^{-x} + e^{-y}) + e^{-x-y}) = e^{-x-y}; \ x > 0; \ y > 0$$

**Question:** Find following probabilities

$$\text{Pr}(X \leq 2, 1 < Y \leq 10) = F(2, 10) = 1 - e^{-2} - e^{-10} + e^{-12}$$

$$\text{Pr}(X \leq 3, 1 < Y \leq 2) = F(x, y) = \int_{0}^{2} \int_{0}^{1} e^{-x-y} \, dy \, dx = \int_{0}^{2} -e^{-y} \bigg|_{0}^{1} \, dx = (e^{-1} - e^{2}) \int_{0}^{2} e^{-x} \, dx = (e^{-1} - e^{2})(-e^{-1}) = (e^{-1} - e^{2})(1-e^{-3})$$

**Question:** Find marginal density of y and x

$$f(y) = \int_{0}^{\infty} e^{-x-y} \, dx = e^{-y} \int_{0}^{\infty} e^{-x} \, dx = e^{-y}(-e^{-\infty}) = e^{-y}; \ y > 0$$

$$f(x) = \int_{0}^{\infty} e^{-x-y} \, dy = e^{-x} \int_{0}^{\infty} e^{-y} \, dy = e^{-x}(-e^{-\infty}) = e^{-x}; \ x > 0$$

**Question:** Find conditional density of X given Y

$$f(x|y) = \frac{e^{-x-y}}{e^{-y}} = e^{-x}; \ x > 0$$

**Question:** Are X and Y independent?

Yes as

$$f(x|y) = \frac{e^{-x-y}}{e^{-y}} = e^{-x} = f(x)$$

$$f(x, y) = e^{-x-y} = e^{-x}e^{-y}$$