CHAPTER 10-Part2
ESTIMATION THEORY

Properties of estimators

**DEFINITION** Unbiased Estimator
Let \( \hat{\theta} \) be the estimator of \( \theta \). If \( E(\hat{\theta}) = \theta \implies \hat{\theta} \) is unbiased estimator of \( \theta \).

**DEFINITION** The bias \( b(\theta) \) is \( b(\theta) = E(\hat{\theta}) - \theta \)

**DEFINITION** \( \hat{\theta} \) is said to be asymptotically unbiased if \( \lim_{n \to \infty} b(\theta) = 0 \)

**Example**
Let \( X_1, X_2, \ldots, X_n \) be a random sample having \( \text{Poi}(\theta) \), where \( \theta \) is unknown. Show if the MLE of \( \theta \) is unbiased estimator.

\[
\hat{\theta}_{\text{MLE}} = \frac{\sum x_i}{n} = \bar{X}
\]

\[
E[\hat{\theta}_{\text{MLE}}] = \theta
\]

\[
E[\bar{X}] = E[\frac{\sum X_i}{n}] = \frac{1}{n} \sum E[X_i] = \frac{1}{n} \sum \theta = n\bar{X} = \theta
\]

UNBIASED!

**Example**
Let \( X_1, X_2, \ldots, X_n \) be a random sample having \( \text{N}(\theta_1, \theta_2) \), where \( \theta_1, \theta_2 \) are unknown. Show if the MLEs of \( \theta = (\theta_1, \theta_2) \) are unbiased estimators.

\[
\hat{\theta}_{1\text{MLE}} = \frac{\sum x_i}{n} = \bar{X}; \quad \hat{\theta}_{2\text{MLE}} = \frac{\sum (x_i - \bar{X})^2}{n}
\]

\[
E[\hat{\theta}_{1\text{MLE}}] = \theta_1
\]

\[
E[\bar{X}] = E[\frac{\sum X_i}{n}] = \frac{1}{n} \sum E[X_i] = \frac{1}{n} \sum \theta = n\bar{X} = \theta \quad \cdot \quad \hat{\theta}_{1\text{MLE}} \text{ is UNBIASED!}
\]
\[ E[\hat{\theta}_{MLE}] = \hat{\theta} \]
\[ E[\hat{\theta}_{MLE}] = E\left[ \frac{1}{n} \sum (x_i\ - \bar{X})^2 \right] = \frac{(n-1)}{n} E\left[ \frac{1}{n-1} \sum (x_i\ - \bar{X})^2 \right] = \frac{\hat{\theta}^2}{n} E\left[ \frac{1}{n-1} s^2 \right] = \frac{(n-1)\hat{\theta}^2}{n} \]

\( \hat{\theta}_{MLE} \) is not UNBIASED!

Bias:
\[ b(\theta) = E[\hat{\theta}_{MLE}] - \theta = \frac{\theta}{n} - \theta = -\frac{\theta}{n} \]

Asymptotically unbiased?
\[ \lim_{n \to \infty} b(\theta) = \lim_{n \to \infty} \left( -\frac{\theta}{n} \right) = 0 \quad \text{YES!} \]

Example
Let \( X_1, X_2 \ldots X_n \) be a random sample having \( U(0, \theta) \), where \( \theta \) is unknown.

Show if the MLE of \( \theta \) is an unbiased estimator.

\( \hat{\theta}_{MLE} = Y_n \)

\[ E[\hat{\theta}_{MLE}] = \theta \implies E[Y_n] = \int_0^\theta y_n g(y_n) dy_n \]

\[ g(y_n) = n[F_X(y_n)]^{n-1} f_X(y_n) = \frac{n}{\theta} \left[ \frac{y_n}{\theta} \right]^{n-1}; \quad 0 < y_n < \theta \]

\[ E[Y_n] = \int_0^\theta y_n n \left[ \frac{y_n}{\theta} \right]^{n-1} dy_n = \frac{n}{n+1} \theta \neq \theta \]

IS NOT UNBIASED!

Bias:
\[ b(\theta) = E[\hat{\theta}] - \theta = \frac{n\theta}{n+1} - \theta = -\frac{\theta}{n+1} \]

Asymptotically unbiased?
\[ \lim_{n \to \infty} b(\theta) = \lim_{n \to \infty} \left( -\frac{\theta}{n+1} \right) = 0 \quad \text{YES!} \]
Efficient estimators

Theorem: If \( \hat{\theta} \) is the unbiased estimator of \( \theta \) and if the following equality holds
\[
\text{Var}(\hat{\theta}) = \frac{1}{nE\left[\left(\frac{\partial \log f(x; \theta)}{\partial \theta}\right)^2\right]},
\]
then \( \hat{\theta} \) is called a Minimum Variance Unbiased Estimator (MVUE).

Definition: Cramér-Rao Lower Bound on Variance (CRLB)

\[
\text{Var}(\hat{\theta}) \geq \frac{1}{nE\left[\left(\frac{\partial \log f(x; \theta)}{\partial \theta}\right)^2\right]}
\]

\[
\text{Var}(\hat{\theta}) \geq \frac{-1}{nE\left[\frac{\partial^2 \log f(x; \theta)}{\partial \theta^2}\right]}
\]

Definition: Relative Efficiency

Let \( \hat{\theta}_1, \hat{\theta}_2 \) be the estimators of \( \theta_1, \theta_2 \). If \( \text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2) \) or \( \frac{\text{Var}(\hat{\theta}_1)}{\text{Var}(\hat{\theta}_2)} < 1 \), then \( \hat{\theta}_1 \) is relatively more efficient than \( \hat{\theta}_2 \).

Relative efficiency with respect to CRLB is

\[
\text{eff}(\hat{\theta}) = \frac{\text{CRLB}}{\text{Var}(\hat{\theta})} \leq 1
\]

Definition: Asymptotic Efficiency

Let \( \hat{\theta}_1, \hat{\theta}_2 \) be the estimators of \( \theta_1, \theta_2 \). If \( \lim_{n \to \infty} \frac{\text{Var}(\hat{\theta}_1)}{\text{Var}(\hat{\theta}_2)} = 1 \), then \( \hat{\theta}_1 \) is asymptotically more efficient than \( \hat{\theta}_2 \).

Definition: Mean Squared Error (MSE)

Let \( \hat{\theta} \) be the estimator of \( \theta \). The Mean Squared Error is

\[
E[(\hat{\theta} - \theta)^2] = \text{Var}(\hat{\theta}) + (b(\theta))^2.
\]
Example
Let $X_1, X_2, \ldots, X_n$ be a random sample having $U(0, \theta)$, where $\theta$ is unknown.

Show if the MLE of $\hat{\theta} = \frac{(n+1)}{n} Y_n$ is an unbiased estimator and compare the CRLB of the density with the variance of the estimator.

\[
E[\hat{\theta}] = \theta \Rightarrow E[\frac{(n+1)}{n} Y_n] = \frac{(n+1)}{n} E[Y_n] = \frac{(n+1)}{n} \int_0^\theta y_n g(y_n) dy_n
\]

\[
g(y_n) = n[F_X(y_n)]^{n-1} f_x(y_n) = n \left( \frac{y_n}{\theta} \right)^{n-1} \quad ; \quad 0 < y_n < \theta
\]

\[
\left( \frac{n+1}{n} \right) E[Y_n] = \left( \frac{n+1}{n} \right) \int_0^\theta y_n \left( \frac{y_n}{\theta} \right)^{n-1} dy_n = \left( \frac{n+1}{n} \right) \frac{n}{n+1} \theta = \theta
\]

UNBIASED!

\[
Var[\frac{(n+1)}{n} Y_n] = \left( \frac{n+1}{n} \right)^2 Var[Y_n]
\]

\[
E[Y_n^2] = \int_0^\theta y_n^2 \left( \frac{y_n}{\theta} \right)^{n-1} dy_n = \frac{n}{n+2} \theta^2
\]

\[
Var[Y_n] = E[Y_n^2] - (E[Y_n])^2
\]

\[
Var[\hat{\theta}] = \left( \frac{n+1}{n} \right)^2 Var[Y_n] = \left( \frac{n+1}{n} \right)^2 \frac{n}{(n+1)^2(n+2)} \theta^2 = \frac{n}{n(n+2)} \theta^2
\]

Cramer-Rao Lower Bound -CRLB:

\[
\frac{d}{d\theta} \log(f(x; \theta)) = \frac{d}{d\theta} \log\left(\frac{1}{\theta}\right) = -\frac{d}{d\theta} \log(\theta) = -\frac{1}{\theta}
\]

\[
\frac{d^2}{d\theta^2} \log(f(x; \theta)) = \frac{1}{\theta^2}
\]

\[
CRLB = \frac{1}{n E \left( \frac{\partial \log f(x; \theta)}{\partial \theta} \right)^2} = \frac{1}{n \left( E \left( \frac{1}{\theta} \right)^2 \right)} = \frac{1}{n} \theta^2 = \frac{\theta^2}{n} \quad OR
\]

\[
CRLB = \frac{-1}{n E \left( \frac{\partial^2 \log f(x; \theta)}{\partial \theta^2} \right)} = \frac{-1}{n \left( \frac{1}{\theta^2} \right)} = \frac{\theta^2}{n}
\]

Compare CRLB with the Variance of the estimator
\[ \frac{CRLB}{\text{Var}[\hat{\theta}]} = \frac{\frac{\theta^2}{n}}{\frac{\theta^2}{n(n+2)}} = n + 2 > 1 \text{ NOT EFFICIENT!} \]

Asymptotic Efficiency
\[ \lim_{n \to \infty} \frac{CRLB}{\text{Var}[\hat{\theta}]} = \lim_{n \to \infty} n + 2 \to \infty \text{ NOT ASYMPTOTICALLY EFFICIENT!} \]

**DEFINITION** Consistent Estimator
Any statistics that converges stochastically to a parameter \( \theta \) is called consistent estimator of that parameter. That is, \( \text{Var}(\hat{\theta}) \to 0 \).