Review Exercises

1. Let $X$ and $Y$ have the p.d.f. $f(x, y) = 1$, $0 < x < 1, 0 < y < 1$, zero elsewhere. Find the p.d.f. of the product $Z = XY$.

2. Let $X$ have the p.d.f. $f(x) = (x + 2)/18$, $-2 < x < 4$, zero elsewhere.

   Find $E[X], E[(X + 2)^3], E[6X - 2(X + 2)^3]$.

3. Let $f(x) = 2x$, $0 < x < 1$, zero elsewhere, be the p.d.f. of $X$.
   
   a. Compute $E[\sqrt{X}]$.
   
   b. Find the distribution function and the p.d.f. of $Y = \sqrt{X}$ and $E[Y]$.

4. Find the mean and the variance of the distribution that has the distribution function

   $$F(x) = \begin{cases} 
   0 & \text{if } x < 0, \\
   \frac{x}{8} & \text{when } 0 \leq x < 2, \\
   \frac{x^2}{16} & \text{when } 2 \leq x < 4, \\
   1 & \text{if } x \geq 4.
   \end{cases}$$

5. Let $f(x, y) = 21x^2y^3$, $0 < x < y < 1$, zero elsewhere, be the joint p.d.f. of $X$ and $Y$. Find the conditional mean and the variance of $X$ and $Y$.

6. Let $X$ and $Y$ have the joint p.d.f. described as follows:

   $$
   f((1,1)) = \frac{2}{15}, \quad f((1,2)) = \frac{4}{15}, \quad f((1,3)) = \frac{3}{15}, \quad f((2,1)) = \frac{1}{15}, \quad f((2,2)) = \frac{1}{15}, \quad f((2,3)) = \frac{4}{15}
   $$

   and $f(x, y)$ is zero elsewhere. Find the correlation coefficient $\rho$.

7. Let $X_1, X_2, X_3, X_4$ be mutually stochastically independent random variables, each with p.d.f. $f(x) = 3(1 - x)^2$, $0 < x < 1$, zero elsewhere. If $Y$ is the minimum of these variables, find the distribution function and the p.d.f. of $Y$.

8. Let $X$ have a Poisson distribution with $\lambda = 100$. Use Chebychev’s inequality to determine a lower bound for $Pr(75 < X < 125)$.

9. Let $X$ have a gamma distribution with parameters $\alpha$ and $\beta$. Show that $Pr(X \geq 2\alpha\beta) \leq (2/e)^\alpha$.

   **Hint:** $Pr(X \geq a) \leq e^{-at}M(t), \quad 0 < t < h, Pr(X \leq a) \leq e^{-at}M(t), \quad -h < t < 0$.

10. Let $X$ and $Y$ have the p.d.f. $f(x, y) = \frac{xy}{36}$, $x = 1, 2, 3, \quad y = 1, 2, 3$, zero elsewhere. Find first the joint p.d.f. of $Z = XY$ and $T=Y$, and then marginal p.d.f. of $Z$.

11. Let $Y_1 < Y_2 < Y_3$ denote the order statistics of a random sample of size 3 from $U(0,1)$ distribution. Let $Z = \frac{Y_1 + Y_3}{2}$ be the midrange of the sample. Find the p.d.f. of $Z$.

12. Let $Z_n$ be $\chi^2_n$ and let $W_n = \frac{Z_n}{n}$. Find the limiting distribution of $W_n$.

13. The Pareto distribution is frequently used as a model in study of incomes and has the distribution function $F(x; \theta_1, \theta_2) = 1 - \left(\frac{\theta_2}{x}\right)^{\theta_1}$, $\theta_2 \leq x$, zero elsewhere, $\theta_2 > 0, \theta_1 > 0$.

   If a random sample of $n$ is taken from Pareto distribution, find the maximum likelihood estimators of $\theta_1$ and $\theta_2$. 