Problem 1 We consider weekly observations of the MSCI World stock market index from January 1990 to May 2009 (T = 1012 observations). Returns are calculated as
\[ r_t = 100 \times \log(I_t/I_{t-1}), \tag{1} \]
where \( I_t \) is the index level at time \( t \). Returns are calculated from Wednesday to Wednesday.

Several properties of the index and the index returns are displayed in Table 1 and Figures 1 and 2.

(a) Describe what you observe in Table 1 and Figures 1 and 2 insofar as it is typical for many financial return series.

<table>
<thead>
<tr>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.051</td>
<td>4.81</td>
<td>-0.79</td>
<td>8.06</td>
</tr>
</tbody>
</table>

“Skewness” denotes the moment–based coefficient of skewness, \( \gamma = m_3/m_2^{3/2} \), and “Kurtosis” the moment–based coefficient of kurtosis, \( \kappa = m_4/m_2^2 \), where \( m_i = T^{-1} \sum_t (r_t - \overline{r})^i \), \( i = 2, 3, 4 \), and \( \overline{r} = T^{-1} \sum_t r_t \).

Problem 2 Explain the Jarque–Bera test for normality. For the return data considered in Problem 1, perform the test at a significance level of \( \alpha = 0.05 \). (Table 2 may be helpful in this regard.)

\(^1\)For more information on the index, see, e.g., http://en.wikipedia.org/wiki/MSCI_World.
**Problem 3** Consider a stationary Gaussian ARMA(7,3) process,

\[ Y_t = \phi_0 + \sum_{i=1}^{7} \phi_i Y_{t-i} + \epsilon_t + \sum_{i=1}^{3} \theta_i \epsilon_{t-i}, \quad \epsilon_t \overset{iid}{\sim} \text{Normal}(0, \sigma^2). \]

Find the kurtosis of the unconditional distribution of \( Y_t \). (Hint: The kurtosis of the iid Gaussian innovation \( \epsilon_t \) is 3).

**Problem 4** (kurtosis of GARCH(1,1) with nonnormal innovations) Consider the GARCH(1,1) process,

\[
\begin{align*}
\epsilon_t &= \sigma_t \eta_t, \\
\sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2,
\end{align*}
\]

where \( \eta_t \) is distributed iid with zero mean, unit variance (i.e., \( E(\eta_t^2) = 1 \)), and kurtosis (fourth moment) \( E(\eta_t^4) = \kappa_4 \).

(i) Find the condition for the existence of a finite fourth moment of the process.

(ii) Calculate the unconditional kurtosis of the GARCH process (2)–(3).

**Problem 5** Consider the following model for the returns, \( r_t \), of a stock market index:

\[
\begin{align*}
r_t &= \mu + \delta \sigma_t + \epsilon_t, \\
\epsilon_t &= \eta_t \sigma_t, \quad \eta_t \overset{iid}{\sim} \text{N}(0, 1) \\
\sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2.
\end{align*}
\]

Is this an economically plausible specification for the conditional mean return, as specified by (4)? What is the expected sign of parameter \( \delta \)? What would you expect the constant \( \mu \) in (4) to be (consider what happens if \( \sigma_t^2 = 0 \))?
**Problem 6** Consider the GARCH(1,1) process

\[ \epsilon_t = \eta_t \sigma_t, \quad \eta_t \overset{iid}{\sim} N(0, 1) \]
\[ \sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \]

We want to calculate multi-step ahead variances. That is, for \( \tau \geq 1 \), find

\[ \text{Var}(\epsilon_t + \tau | I_t) = E(\epsilon_t^2 | I_t), \tag{7} \]

where \( I_t = \{ \epsilon_s : s \leq t \} \) is the information set at time \( t \). If we have a model for daily returns and we are interested in, for example, weekly or monthly returns, we also need quantities of the form

\[ \text{Var}(\epsilon_{t+1} + \cdots + \epsilon_{t+\tau} | I_t). \tag{8} \]

**Problem 7** Suppose that the return \( r_t \) of your portfolio is generated by

\[ r_t = 0.025 + \epsilon_t \]
\[ \epsilon_t = \eta_t \sigma_t, \quad \eta_t \overset{iid}{\sim} N(0, 1) \]
\[ \sigma_t^2 = 0.025 + 0.075 \epsilon_{t-1}^2 + 0.9 \sigma_{t-1}^2. \]

Your current estimate for \( \sigma_t^2 \) is its unconditional expectation. Unfortunately, however, due to unpredictable adverse market conditions, your portfolio suffers from an unusually large negative shock, so that \( r_t = -4.75 \). Calculate the 1% Value–at–Risk for period \( t + 1 \).
Figure 1: MSCI index levels.
Figure 2: MSCI index returns.
Table 2: Quantiles of the $\chi^2$ distribution.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$z_{0.9}$</th>
<th>$z_{0.95}$</th>
<th>$z_{0.975}$</th>
<th>$z_{0.99}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.7055</td>
<td>3.8415</td>
<td>5.0239</td>
<td>6.6349</td>
</tr>
<tr>
<td>2</td>
<td>4.6052</td>
<td>5.9915</td>
<td>7.3778</td>
<td>9.2103</td>
</tr>
<tr>
<td>3</td>
<td>6.2514</td>
<td>7.8147</td>
<td>9.3484</td>
<td>11.3449</td>
</tr>
<tr>
<td>4</td>
<td>7.7794</td>
<td>9.4877</td>
<td>11.1433</td>
<td>13.2767</td>
</tr>
<tr>
<td>5</td>
<td>9.2364</td>
<td>11.0705</td>
<td>12.8325</td>
<td>15.0863</td>
</tr>
<tr>
<td>6</td>
<td>10.6446</td>
<td>12.5916</td>
<td>14.4494</td>
<td>16.8119</td>
</tr>
<tr>
<td>7</td>
<td>12.0170</td>
<td>14.0671</td>
<td>16.0128</td>
<td>18.4753</td>
</tr>
<tr>
<td>8</td>
<td>13.3616</td>
<td>15.5073</td>
<td>17.5345</td>
<td>20.0902</td>
</tr>
<tr>
<td>10</td>
<td>15.9872</td>
<td>18.3070</td>
<td>20.4832</td>
<td>23.2093</td>
</tr>
</tbody>
</table>

$\nu$ denotes the degrees of freedom of the $\chi^2$ distribution, and $z_\alpha$ is the $\alpha$-Quantile, that is, $z_\alpha$ is such that

$$\int_0^{z_\alpha} \chi^2(z; \nu)dz = \alpha, \quad (9)$$

where $\chi^2(z; \nu)$ is the density function of a $\chi^2$ random variable with $\nu$ degrees of freedom.