Linear difference equations appear in various situations in time series analysis. Consider the situation where a variable at time $t$, $y_t$, is a linear function of its last $p$ realizations; the equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p}$$

is a linear homogeneous difference equation of order $p$.

Another frequently used representation of (1) makes use of the lag operator, $L$, which shifts the time index backward by one unit, that is,

$$Ly_t = y_{t-1}.$$  \hspace{1cm} (3)

Repeated application of the lag operator is denoted with the appropriate exponent, that is, for example,

$$L^3 y_t = LLLy_t = y_{t-3},$$ \hspace{1cm} (4)

and, more generally,

$$L^q y_t = y_{t-q}.$$ \hspace{1cm} (5)

We can then write (1) as

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)y_t = 0,$$ \hspace{1cm} (6)

or, in short,

$$\phi(L)y_t = 0,$$ \hspace{1cm} (7)

\footnote{The nonhomogeneous equation would read

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + c_t,$$ \hspace{1cm} (2)

with $c_t$ being a function of time, $t$ (the simplest form being a constant: $c_t = c$). We shall only consider homogeneous equations.}
where $\phi(L)$ denotes the lag polynomial

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p.$$  \hspace{1cm} (8)

To solve equations of the form (1), let us first consider the equation of order one, that is,

$$y_t = \phi_1 y_{t-1}.$$  \hspace{1cm} (9)

This has the solution

$$y_t = \phi_1^t y_0,$$  \hspace{1cm} (10)

where $y_0$ is the value of $y$ at time $t = 0$ (the initialization). Clearly the behavior of the equation depends on the coefficient $\phi_1$; in particular,

$$\lim_{t \to \infty} y_t = 0 \quad \text{for any } y_0 \text{ if and only if } |\phi_1| < 1.$$  \hspace{1cm} (11)

This is the stability condition for the first–order equation (9).

The dynamics of higher–order equations are governed by the roots of the characteristic equation associated with (1), given by

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \cdots - \phi_{p-1} \lambda - \phi_p = 0.$$  \hspace{1cm} (12)

A polynomial of order $p$ in general has $p$ roots (counting multiplicities), i.e., values satisfying (12). Assume for simplicity that all roots are distinct and denote these roots by $\lambda_1, \lambda_2, \ldots, \lambda_p$. Then the general solution of (1) is\footnote{Some modifications have to be made when there are multiple roots; see, e.g., Hamilton, Chapter 1. The qualitative conclusions and the stability condition (14) remain unchanged, however.}

$$y_t = c_1 \lambda_1^t + c_2 \lambda_2^t + \cdots + c_p \lambda_p^t,$$  \hspace{1cm} (13)

where the coefficients $c_i, i = 1, \ldots, p$, have to determined from $p$ initial conditions (that is, prescribed values for the e.g. first $p$ values of the process). It is apparent that the stability condition for the $p$th order equation is

$$\max\{|\lambda_i| : 1 \leq i \leq p\} < 1,$$  \hspace{1cm} (14)

i.e., all roots of the characteristic equation are smaller than one in modulus. The smaller the largest root is in modulus, the faster the solution tends to the origin. Occasionally,
the condition is also stated in terms of the reverse characteristic polynomial (cf. Equation (8))
\[ \phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = z^p (z^{-p} - \phi_1 z^{-p+1} - \cdots - \phi_{p-1} z^{-1} - \phi_p) = 0, \quad (15) \]
the roots of which are the inverses of those of (12), i.e., the roots of (15) have to be larger than one in modulus (have to be outside the unit circle).

For illustration, we consider equations of order two. The solution of
\[ y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} \quad (16) \]
is
\[ y_t = c_1 \lambda_1^t + c_2 \lambda_2^t, \quad (17) \]
where, assuming that the roots are distinct \((\phi_1^2 \neq -4\phi_2)\),
\[ \lambda_{1/2} = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}. \]
Consider equation
\[ y_t = 1.4 y_{t-1} - 0.48 y_{t-2} \quad (18) \]
with initial conditions \(y_0 = 1\) and \(y_1 = 0.95\). The characteristic equation
\[ \lambda^2 - 1.4\lambda + 0.48 = 0 \quad (19) \]
has roots
\[ \lambda_1 = 0.8, \quad \lambda_2 = 0.6, \quad (20) \]
so the equation is stable and converges to the origin as \(t\) becomes large. The solution is of the form
\[ y_t = c_1 0.8^t + c_2 0.6^t. \quad (21) \]
The constants \(c_1\) and \(c_2\) have to be determined from the initial conditions, i.e., by solving the system
\[ c_1 + c_2 = 1, \quad 0.8 \times c_1 + 0.6 \times c_2 = 0.95, \quad (22) \]
which results in \(c_1 = 1.75\) and \(c_2 = -0.75\), so the solution is
\[ y_t = 1.75 \times 0.8^t - 0.75 \times 0.6^t, \quad (23) \]
Figure 1: Solution of Equation (18) with initial conditions \( y_0 = 1 \) and \( y_1 = 0.95 \). which is shown in Figure 1.

An oscillatory behavior appears when the characteristic equation has complex roots. For example, equation

\[
y_t = y_{t-1} - 0.89y_{t-2}
\]

has characteristic equation \( \lambda^2 - \lambda + 0.89 = 0 \) with roots

\[
\lambda_{1/2} = 0.5 \pm \sqrt{0.25 - 0.89} = 0.5 \pm i\sqrt{0.64} = 0.5 \pm i0.8
\]

To represent the general solution, we write the complex roots in polar coordinate form

\[
z = x \pm iy = R(\cos \theta \pm i \sin \theta),
\]

where \( R = |z| = \sqrt{x^2 + y^2} = \sqrt{0.5^2 + 0.8^2} = \sqrt{0.89} \) is the modulus of the complex number and \( \theta = \arccos(x/R) = 1.0122 \) (measured in radians). De Moivre’s formula states that

\[
(cos \theta + i \sin \theta)^t = \cos(t \theta) + i \sin(t \theta),
\]
and the general solution of (24) is

$$y_t = R^t (c_1 \cos(t \theta) + c_2 \sin(t \theta)),$$

with coefficients $c_1$ and $c_2$ again being determined from the initial conditions. For example, if we require $y_0 = 1$ and $y_1 = 0.53$, then $c_1 = 1$ since $\cos 0 = 1$ and $\sin 0 = 0$, and so $c_2$ is obtained from

$$0.53 = R [\cos \theta + c_2 \sin \theta] \Rightarrow c_2 = \frac{0.53 - R \cos \theta}{R \sin \theta} = 0.0375,$$

hence the solution is

$$y_t = (\sqrt{0.89})^t [\cos(t \times 1.0122) + 0.0375 \sin(t \times 1.0122)],$$

which is shown in Figure 2. The equation is stable and the solution tends to the origin since the modulus $|z| = R = \sqrt{0.89} < 1$. 

Figure 2: Solution of Equation (24) with initial conditions $y_0 = 1$ and $y_1 = 0.53$. 