Summer Term 2009

Albert-Ludwigs-Universität Freiburg
Empirische Forschung und Okonometrie

Stochastic Time Series Analysis-Part 1

Moving Average(q) Model: MA(q)
MA(q) Model

- Linear Process:  
  \[ X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \ldots + \theta_q Z_{t-q} + \ldots \]  
  \[ X_t = \sum_{i=0}^{\infty} \theta_i Z_{t-i} \]  
  \[ \theta_q = 1 \]

- White Noise Process (WN): 
  \{X_t\} is a sequence of i.i.d random variables with zero mean and finite variance \(\sigma^2\).  
  The series is stationary with \(\gamma(t+h,t) = \sigma^2\) for \(h=0\).

White Noise process is a purely random process where all autocorrelation functions for every \(h\) are close to zero.

\[ \nu_t = \frac{1}{3}(w_{t-1} + w_t + w_{t+1}) \]
MA(q) Model

Let \( \{Z_t\} \sim WN(0, \sigma^2) \)

MA(q) is

\[
X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \ldots + \theta_q Z_{t-q} = \sum_{j=0}^{q} \theta_j Z_{t-j}
\]

For \( q = 1 \) \( MA(1) \)

\[
E[Z_t] = 0, \quad V[Z_t] = \sigma^2 (1 + \theta^2)
\]

\[
\gamma_s(t+h, t) = \begin{cases} 
\sigma^2 (1 + \theta^2) & h = 0 \\
\sigma^2 \theta & h = \mp 1 \\
0 & |h| > 1 
\end{cases}
\]

\[
\rho_s(h) = \begin{cases} 
\frac{1}{\theta} & h = 0 \\
\frac{1 + \theta^2}{1 + \theta^2} & h = \mp 1 \\
0 & |h| > 1 
\end{cases}
\]
Invertibility

The estimated values of parameters have certain conditions. \( X_t = Z_t + \theta Z_{t-1} = Z_t + \theta BZ_t = (1 - \theta B)Z_t \)

\[ Z_t = \frac{X_t}{1 + \theta B} = (\frac{1}{1 + \theta B})X_t = \theta(B)Z_t \]

In general, MA(q) is invertible if the roots of \( \theta(B) \) lie in unit circle

\[ X_t = Z_t + \theta Z_{t-1} + \ldots + \theta Z_{t-q} \]

\[ X_t = Z_t + \theta BZ_t + \theta B^2Z_t + \ldots + \theta B^q Z_t \]

\[ X_t = Z_t (1 + \theta B + \theta B^2 + \ldots + \theta B^q) = Z_t \sum_{j=0}^{q} \theta B^j \]

\[ \Rightarrow X_t = \frac{Z_t}{\sum \theta B^j} = (\frac{1}{\sum \theta B^j})Z_t = \theta(B)Z_t \]

Stationary Time Series

- Strictly stationary process: If the joint dist. of \((X_{t1}, \ldots, X_{tn})\) is the same as the joint dist. of \((X_{t1+k}, \ldots, X_{tn+k})\)

- Weakly Stationary process (second order stationary): \{X_t\} is weakly stationary if
  i. \( \mu_s(t) = E(X_t) \) is independent of \( t \).
  ii. \( Cov(X_r, X_s) = \gamma_s(r, s) = E[(X_r - \mu_r)(X_s - \mu_s)] \)
      does not depend on \( t \).
Random Walk

- Random Walk Model:
  Let \( \{S_t, t=1,\ldots,n\} \) be a process with \( S_t = \sum Z_t \) where \( Z_t \) is WN.
  Then, \( E[S_t]=0, \ Var[S_t]=t.\sigma^2 \) and \( \gamma(t+h,t)= t.\sigma^2 \)

- Since \( \gamma(t+h,t)= t.\sigma^2 \) depends on \( t \),
  the series is not stationary.
  However, \( X_t = Z_t - Z_{t-1} \) is stationary.
AR(p) Model and its autocorrelation

Let \( \{Z_t\} \sim WN(0, \sigma^2) \)

AR(p) is

For \( p=1 \) \( \text{AR}(1) \)

\[
X_t = \phi X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + Z_t
\]

\[
E[Z_t] = 0
\]

\[
V[Z_t] = \sigma^2/(1-\phi^2)
\]

\[
\gamma_s(h) = \frac{\sigma^2 \phi^h}{1-\phi^2}
\]

\[
\rho_s(h) = \phi^h
\]
From the output, the behavior of ACF for AR(2) is tails off; the behavior of PACF for AR(2) is cuts off after lag 2.

Causality

The estimated values of parameters have certain conditions. $X_t = \mu + \phi X_{t-1} + Z_t$, where $|\phi| < 1$

$X_t - \mu = \phi(X_{t-2} + Z_{t-1}) + Z_t$

after substitutions

$X_t - \mu = (1 + \phi B + \phi^2 B + \ldots)Z_t$

$X_t - \mu = 1 - \phi B)Z_t = \phi(B)Z_t$

$\Rightarrow X_t = \mu + (\phi B + \phi^2 B + \ldots)Z_t = Z \sum_{j=0}^{\infty} \phi B^j$

In general, AR(p) is causal if the roots of $\Phi(B)$ lie in unit circle

$X_t = Z(1 + \phi B + \phi^2 B + \ldots + \phi^p B^p) = Z \sum_{j=0}^{p} \phi B^j$

$\Rightarrow X_t = \frac{Z_t}{\sum \phi B^j} = \frac{1}{\sum \phi B^j}Z_t = \phi(B)Z_t$
ARMA(p,q) Model

Let \( \{Z_t\} \sim WN(0, \sigma^2) \)

ARMA(p,q) is

\[
X_t = \mu + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}
\]

\[
\phi(B)(X_t - \mu) = \theta(B)Z_t
\]

where

\[
\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p
\]

\[
\theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q
\]

ARMA(1,1)

\[
X_t = \mu + \phi(X_{t-1} - \mu) + \theta Z_{t-1}
\]

\[
\phi(B)(X_t - \mu) = \theta(B)Z_t
\]

where

\[
\phi(B) = 1 - \phi B
\]

\[
\theta(B) = 1 + \theta B
\]

Partial Autocorrelation PACF

- For an AR(p) process PACF, \( \Phi_{hh} \) is the correlation between \( X_t \) and \( X_{t-h} \) controlling the effect of \( X_{t-h-1} \)

- AR(1): \( \Phi_{11} = \Phi = \rho(1) \)

- AR(2): \( \Phi_{11} = \Phi = \rho(1) \)
  \[
  \phi_{22} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = \phi_2
  \]
  \[
  \phi_{hh} = 0 \quad h > 2
  \]
PACF

- **AR(p)**

  \[
  \phi_1 = \rho(1); \quad \phi_2 = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2}; \quad \phi_h = \frac{\rho(h) - \sum_{j=1}^{h-1} \rho(j-1) \phi_h(j-h)}{1 - \sum_{j=1}^{h-1} \rho(j)}, \quad h = 3, 4, \ldots
  \]

  \[
  \phi_h = \phi_{h+1} - \phi_h \phi_{h+1-j}, \quad j = 1, 2, \ldots, h-1
  \]

- **Yule Walker Equations**

  \[
  \gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + \cdots + \phi_p \gamma(h-p) \quad h > 0
  \]

  \[
  \rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2) + \cdots + \phi_p \rho(h-p) \quad h > 0
  \]

Partial Autocorrelation for MA(1) process

- \( \phi_{hh} = \frac{\theta(1-\theta')}{1-\theta'^2} \) for \( h > 0 \)
- \( \phi_{11} = \frac{\theta(1-\theta')}{1-\theta'} \)
- \( \phi_{22} = \frac{\theta' (1-\theta')}{1-\theta'} \)
- \( \phi_{33} = \frac{\theta' (1-\theta')}{1-\theta'} \)

Asymptotic distribution of Partial Autocorrelations

For a causal AR(p) process, the asymptotic distribution

\[
\sqrt{n} \hat{\phi}_{kk} \overset{d}{\rightarrow} N(0,1)
\]
Properties of the ACF and PACF for various ARMA Models

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<tr>
<th>Model</th>
<th>ACF</th>
<th>PACF</th>
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<td>AR(1)</td>
<td>Exponential or oscillatory decay</td>
<td>$a_1 = 0$ for $h &gt; 1$</td>
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<tr>
<td>AR(2)</td>
<td>Exponential or sine wave decay</td>
<td>$a_2 = 0$ for $h &gt; 2$</td>
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<tr>
<td>AR(p)</td>
<td>Exponential or sine wave decay</td>
<td>$a_p = 0$ for $h &gt; p$</td>
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<tr>
<td>MA(1)</td>
<td>$\rho_1 = 0$ for $h &gt; 1$</td>
<td>Dominated by damped exponential</td>
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<tr>
<td>MA(2)</td>
<td>$\rho_1 = 0$ for $h &gt; 2$</td>
<td>Dominated by damped exponential or sine wave</td>
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<td>MA(q)</td>
<td>$\rho_q = 0$ for $h &gt; q$</td>
<td>Dominated by linear combination of damped exponential and/or sine waves</td>
</tr>
<tr>
<td>ARMA(1,1)</td>
<td>Tails off. Exponential decay from lag 1</td>
<td>Tails off. Dominated by exponential decay from lag 1</td>
</tr>
<tr>
<td>ARMA(p,q)</td>
<td>Tails off after $(p+q)$ lags. Exponential and/or sine wave decay after $(p+q)$ lags</td>
<td>Tails off after $(p+q)$ lags. Dominated by damped exponentials and or sine waves after $(p+q)$ lags</td>
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