Financial Data Analysis

GARCH Models, Part II

June 4, 2010
ARCH and GARCH: Brief Review

- Autoregressive conditional heteroskedasticity of order \( q \) (ARCH\((q)\)) of Engle (1982):

\[
\begin{align*}
  r_t & = \mu_t + \epsilon_t \\
  \epsilon_t & = \eta_t \sigma_t, \quad \eta_t \sim \text{N}(0, 1), \\
  \sigma_t^2 & = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2, \\
  \omega & > 0, \quad \alpha_i \geq 0, \quad i = 1, \ldots, q.
\end{align*}
\]

- Problem: Many lags necessary to obtain a satisfactory fit of volatility dynamics.

- A more parsimonious model is provided by the generalized ARCH of orders \( p \) and \( q \) (GARCH\((p, q)\)):

\[
\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2.
\]
• For Example, GARCH(1,1) is ARCH(∞) with geometrically decreasing lag structure,

\[
\sigma_t^2 = \frac{\omega}{1 - \beta_1} + \alpha_1 \sum_{i=1}^{\infty} \beta_i^{i-1} \epsilon_{t-i}^2.
\]

• The unconditional variance (or log–term variance) of the GARCH\((p, q)\) process is

\[
E(\sigma_t^2) = E(\epsilon_t^2) = \frac{\omega}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_i},
\]

provided the (covariance) stationarity condition

\[
\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i < 1
\]

is satisfied.

• To characterize the correlation structure of the squared process, define the prediction error

\[
u_t = \epsilon_t^2 - E(\epsilon_t^2|I_{t-1}) = \epsilon_t^2 - \sigma_t^2.
\] (2)
• \( u_t = \epsilon_t^2 - \sigma_t^2 = (\eta_t^2 - 1)\sigma_t^2 \) is white noise but not strict white noise, since it is uncorrelated but not independent.

• Substituting (3) for \( \sigma_t^2 \) into (1) results in

\[
\epsilon_t^2 = \omega + \sum_{i=1}^{\max\{p,q\}} (\alpha_i + \beta_i) \epsilon_{t-i}^2 - \sum_{i=1}^{p} \beta_i u_{t-i} + u_t,
\]

where \( \alpha_i = 0 \) for \( i > q \) and \( \beta_i = 0 \) for \( i > p \).

• Equation (3) is an ARMA(\( \max\{p, q\}, p \)) representation for the squared process \( \{\epsilon_t^2\} \), which characterizes its autocorrelations.

• For example, for GARCH(1,1),

\[
\text{Corr}(\epsilon_t^2, \epsilon_{t-\tau}^2) = (\alpha_1 + \beta_1)^{\tau-1} \frac{\alpha_1(1 - \alpha_1 \beta_1 - \beta_1^2)}{1 - 2\alpha_1 \beta_1 - \beta_1^2},
\]

the decay is slow if \( \alpha_1 + \beta_1 \) is close to unity, so \( \alpha_1 + \beta_1 \) determines the rate of convergence of the conditional volatility to the long–term volatility \( \omega/(1 - \alpha_1 - \beta_1) \) and measures persistence of volatility.
• The kurtosis of the GARCH(1,1) with normal innovations $\eta_t$ is

\[
\frac{E(\epsilon_t^4)}{E^2(\epsilon_t^2)} = \frac{3(1 - \alpha_1 - \beta_1)(1 + \alpha_1 + \beta_1)}{1 - 3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2} = 3 + \frac{6\alpha_1^2}{1 - 3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2},
\]

provided $3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1$. 
The graph shows the S&P 500 returns and a simulated path of the fitted GARCH model. The S&P 500 returns exhibit volatility clustering, which is a characteristic feature of financial time series. The GARCH model is used to capture this volatility clustering by modeling the conditional variance of the returns. The simulated path of the fitted GARCH model suggests that the model is capable of reproducing the observed volatility patterns in the S&P 500 returns.
Example and Extensions

Consider the GARCH(1,1) specification,

\[ r_t = \mu + \eta_t \sigma_t, \quad \eta_t \overset{iid}{\sim} \text{Normal}(0, 1), \]

\[ \sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2. \]

Table 1: GARCH(1,1) estimates for various stock return series, January 1990 to October 2009 (S&P 500: March 2010)

<table>
<thead>
<tr>
<th>Series</th>
<th>( \hat{\omega} )</th>
<th>( \hat{\alpha}_1 )</th>
<th>( \hat{\beta}_1 )</th>
<th>( \hat{\alpha}_1 + \hat{\beta}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.0077 (0.0017)</td>
<td>0.0655 (0.0067)</td>
<td>0.9284 (0.0072)</td>
<td>0.9939</td>
</tr>
<tr>
<td>DAX</td>
<td>0.0355 (0.0053)</td>
<td>0.0918 (0.0089)</td>
<td>0.8910 (0.0099)</td>
<td>0.9828</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.0113 (0.0025)</td>
<td>0.0856 (0.0081)</td>
<td>0.9059 (0.0087)</td>
<td>0.9915</td>
</tr>
<tr>
<td>CAC 40</td>
<td>0.0290 (0.0054)</td>
<td>0.0851 (0.0085)</td>
<td>0.9001 (0.0097)</td>
<td>0.9852</td>
</tr>
</tbody>
</table>
Diagnostics can be based on the sequence

$$\hat{\eta}_t = \frac{\epsilon_t}{\sigma_t}, \quad t = 1, \ldots, T.$$  

(4)
Table 2: Kurtosis of raw returns and residuals (4)

<table>
<thead>
<tr>
<th></th>
<th>S&amp;P 500</th>
<th>DAX</th>
<th>FTSE</th>
<th>CAC 40</th>
</tr>
</thead>
<tbody>
<tr>
<td>raw returns</td>
<td>12.1307</td>
<td>8.0553</td>
<td>9.6318</td>
<td>7.8069</td>
</tr>
<tr>
<td>residuals (4)</td>
<td>4.8993</td>
<td>9.6475</td>
<td>3.8232</td>
<td>4.9332</td>
</tr>
</tbody>
</table>
Alternative Innovation Distributions

- In view of these results, it appears reasonable to replace the normal distribution of $\eta_t$ in the GARCH(1,1) with a more flexible alternative that allows for conditional leptokurtosis.

- Two of the most popular candidates in this regard are the
  - Student’s $t$
  - Generalized Error Distribution (GED)

- The unit–variance versions of these are given by

$$f(\eta_t; \nu) = \frac{\Gamma \left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2) \sqrt{(\nu - 2)\pi}} \left(1 + \frac{\eta_t^2}{\nu - 2}\right)^{-(\nu+1)/2}, \quad (5)$$

and

$$f(\eta_t; p) = \frac{\lambda p}{2^{1/p+1} \Gamma(1/p)} \exp \left\{-\frac{|\lambda \eta_t|^p}{2}\right\}, \quad (6)$$

where $\lambda = 2^{1/p} \sqrt{\Gamma(3/p)/\Gamma(1/p)}$. 
Covariance Stationarity and Unconditional Variance for General Innovation Distributions

- In the GARCH\((p, q)\),

\[
\begin{align*}
\epsilon_t &= \eta_t \sigma_t \\
\sigma_t^2 &= \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2,
\end{align*}
\]

(7)

to find the unconditional variance, we can take expectations on both sides,

\[
E(\sigma_t^2) = \omega + \sum_{i=1}^{q} \alpha_i E(\epsilon_{t-i}^2) + \sum_{i=1}^{p} \beta_i E(\sigma_{t-i}^2).
\]

- If the innovations \(\eta_t\) in (7) have unit variance, it follows that

\[
E(\epsilon_t^2) = E(\sigma_t^2) = \frac{\omega}{1 - \sum_i \alpha_i - \sum_i \beta_i},
\]

(8)
provided the second–order stationarity condition

\[ \sum_i \alpha_i + \sum_i \beta_i < 1 \]  \hspace{1cm} (9)

is satisfied.

• However, non–normal densities are not always applied in standardized (unit–variance) form.

• For example, the “conventional” Student’s \( t \) is also often used and has density

\[ f(\eta_t) = \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\Gamma(\nu/2)\sqrt{\nu\pi}} \left( 1 + \frac{\eta_t^2}{\nu} \right)^{-(\nu+1)/2}, \]

which has (for \( \nu > 2 \))

\[ \mathbb{E}(\eta_t^2) = \frac{\nu}{\nu - 2}. \]

• If, in general, \( \mathbb{E}(\eta_t^2) = \kappa_2 \), then (9) and (8) become

\[ \kappa_2 \sum_i \alpha_i + \sum_i \beta_i < 1, \]
and

\[ E(\epsilon_t^2) = \kappa_2 E(\sigma_t^2) = \frac{\kappa_2 \omega}{1 - \kappa_2 \sum_i \alpha_i - \sum_i \beta_i}, \]

respectively.
Table 3: GARCH(1,1) estimates for various stock return series, January 1990 to October 2009

<table>
<thead>
<tr>
<th>Series</th>
<th>Student's $t$</th>
<th>GED</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\omega}$</td>
<td>$\hat{\alpha}_1$</td>
</tr>
<tr>
<td>CAC 40</td>
<td>0.0197 (0.0048)</td>
<td>0.0751 (0.0081)</td>
</tr>
<tr>
<td>DAX</td>
<td>0.0154 (0.0039)</td>
<td>0.0852 (0.0092)</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.0103 (0.0025)</td>
<td>0.0797 (0.0084)</td>
</tr>
</tbody>
</table>
Table 4: Maximized log–likelihood values

<table>
<thead>
<tr>
<th></th>
<th>CAC 40</th>
<th>DAX</th>
<th>FTSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>-8088.5</td>
<td>-8180.9</td>
<td>-6798.8</td>
</tr>
<tr>
<td>Student’s t</td>
<td>-8032.5</td>
<td>-8048.2</td>
<td>-6768.2</td>
</tr>
<tr>
<td>GED</td>
<td>-8048.6</td>
<td>-8085.1</td>
<td>-6779.0</td>
</tr>
</tbody>
</table>

Differences in log–likelihood

<table>
<thead>
<tr>
<th></th>
<th>Student’s t–Normal</th>
<th>GED–Normal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>56.0047</td>
<td>39.8959</td>
</tr>
<tr>
<td></td>
<td>132.6939</td>
<td>95.7972</td>
</tr>
<tr>
<td></td>
<td>30.6056</td>
<td>19.8580</td>
</tr>
</tbody>
</table>
IGARCH and EWMA

- The finding that often $\alpha_1 + \beta_1 \approx 1$ has led to the suggestion of imposing the restriction
  \[ \alpha_1 + \beta_1 = 1, \]
  which is referred to as IGARCH(1,1) (integrated GARCH), since there is a “unit root” in the GARCH polynomial.

- However, the analogy to integrated (unit root) processes is rather weak.

- In particular, IGARCH(1,1) processes are (strictly) stationary, although their second moment does not exist.

- Nelson (1990) has shown that the GARCH(1,1) is strictly stationary if
  \[ \mathbb{E}[\log(\alpha_1 \eta_t^2 + \beta_1)] < 0. \]

- By Jensen’s inequality, for the IGARCH(1,1),
  \[ \mathbb{E}[\log(\alpha_1 \eta_t^2 + \beta_1)] < \log \mathbb{E}(\alpha_1 \eta_t^2 + \beta_1) = \log 1 = 0. \]
• $\alpha_1 + \beta_1$ may be even larger than unity. For example, the ARCH(1) process with $\alpha_1 = 3$ is stationary, although extremely fat–tailed.

• A special case of an IGARCH model (with zero intercept) is the exponentially weighted moving average (EWMA) popularized by RiskMetrics of J.P. Morgan, which is

$$
\sigma^2_t = (1 - \lambda) \sum_{i=0}^{\infty} \lambda^i \epsilon^2_{t-1-i} = (1 - \lambda)\epsilon^2_{t-1} + \lambda \sigma^2_{t-1}, \quad 0 < \lambda < 1, \quad (10)
$$

with $\lambda$ fixed at 0.94 for daily data.
In the Gaussian GARCH model, series

$$\hat{\eta}_t = \frac{\epsilon_t}{\sigma_t}, \quad t = 1, \ldots, T,$$

should be iid standard normally distributed.

Similarly, in a $t$ or GED GARCH model, (11) should follow an iid standard $t$ or GED distribution.

A general and simple technique to generate iid standard normal residuals is as follows.

Calculate the series

$$u_t = F(r_t|I_{t-1}), \quad t = 1, \ldots, T,$$  

(12)
where \( F(\cdot | I_{t-1}) \) is the conditional cumulative distribution function (cdf) of the return \( r_t \) implied by the model under consideration, based on information up to time \( t - 1, I_{t-1} \).

- If the model is correctly specified, (12) is a series of iid uniform(0,1) variables. This is also known as the Rosenblatt transform.

- Subsequently, apply a second transformation, namely,

\[
\{z_t\} = \Phi^{-1}(\{u_t\}),
\]

where \( \Phi^{-1} \) is the inverse of the standard normal cdf.

- If the model is correctly specified, (13) is a sequence of iid standard normal variables.

- This allows the use of standard and simple normality tests for correct specification of (conditional) skewness and kurtosis.
• Let \( \hat{s} \) and \( \hat{\kappa} \) be the sample skewness and kurtosis, respectively, i.e.,

\[
\hat{s} = \frac{T^{-1} \sum_t (z_t - \bar{z})^3}{\left\{ T^{-1} \sum_t (z_t - \bar{z})^2 \right\}^{3/2}}, \quad \hat{\kappa} = \frac{T^{-1} \sum_t (z_t - \bar{z})^4}{\left\{ T^{-1} \sum_t (z_t - \bar{z})^2 \right\}^2}.
\]

• Under normality,

\[
\hat{s} \stackrel{asy}{\sim} \text{Normal}(0, 6/T), \quad \hat{\kappa} \stackrel{asy}{\sim} \text{Normal}(3, 24/T), \quad (14)
\]

so

\[
T \hat{s}^2/6 \stackrel{asy}{\sim} \chi^2(1), \quad T(\hat{\kappa} - 3)^2/24 \stackrel{asy}{\sim} \chi^2(1), \quad (15)
\]

and the Jarque–Bera test

\[
JB = T \hat{s}^2/6 + T(\hat{\kappa} - 3)^2/24 \stackrel{asy}{\sim} \chi^2(2). \quad (16)
\]

• We can also test for absence of autocorrelation, zero mean and unit variance by means of likelihood ratio tests based on the Gaussian likelihood.
Economic Evaluation: Value–at–Risk (VaR)

- For a given model, the VaR at level $\xi$ for period $t$, denoted by $\text{VaR}_t(\xi)$, is implicitly defined by

$$
\widehat{F}(\text{VaR}_t(\xi)|I_{t-1}) = \xi.
$$

- A violation or hit is said to occur at time $t$ if

$$
r_t < \text{VaR}_t(\xi).
$$

- To test the models' suitability for calculating accurate ex–ante VaR measures, we define the binary sequence

$$
I_t = \begin{cases} 
1, & \text{if } r_t < \text{VaR}_t \\
0, & \text{if } r_t \geq \text{VaR}_t.
\end{cases}
$$  \hspace{1cm} (17)
• Then the empirical shortfall probability is

\[ \hat{\xi} = \frac{x}{T}, \quad \text{where } x = \sum_{t=1}^{T} I_t, \]

is the number of observed violations.

• For a correctly specified VaR model, we expect \(100 \times \xi\)% of the observed return values not to fall below the respective VaR threshold.

• To assess whether the empirical shortfall probability, \(\hat{\xi}\), is statistically indistinguishable from the nominal shortfall probability, \(\xi\), we use the likelihood ratio test

\[
\text{LRT}_{\text{VaR}} = -2 \{x \log(\xi/\hat{\xi}) + (T - x) \log[(1 - \xi)/(1 - \hat{\xi})]\} \overset{asy}{\sim} \chi^2(1). \quad (18)
\]

• If \(\hat{\xi}\) is significantly less (higher) than \(\xi\), then the model tends to overestimate (underestimate) the risk of the position.
One–step–ahead predictive densities

- First estimate the models over the (approximately) first ten years of data, i.e., the first 2500 observations.

- Then update the parameters (approximately) every month (i.e., 20 trading days) employing a moving window of data, i.e., using the most recent 2500 observations in the sample.

- We get, for each model and series, 2480 one–step–ahead predictive densities for the period January 2000 to October 2009.
Table 5: GARCH(1,1) density forecasts based on (13)

<table>
<thead>
<tr>
<th>Series</th>
<th>mean</th>
<th>var.</th>
<th>skewness</th>
<th>kurtosis</th>
<th>JB</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>-0.0562***</td>
<td>1.0229</td>
<td>-0.304***</td>
<td>4.014***</td>
<td>144.5***</td>
</tr>
<tr>
<td>DAX</td>
<td>-0.0567***</td>
<td>1.0249</td>
<td>-0.317***</td>
<td>3.945***</td>
<td>133.7***</td>
</tr>
<tr>
<td>FTSE</td>
<td>-0.0517**</td>
<td>1.0221</td>
<td>-0.354***</td>
<td>3.746***</td>
<td>109.3***</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Series</th>
<th>mean</th>
<th>var.</th>
<th>skewness</th>
<th>kurtosis</th>
<th>JB</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>-0.0569***</td>
<td>1.0162</td>
<td>-0.221***</td>
<td>3.327***</td>
<td>31.20***</td>
</tr>
<tr>
<td>DAX</td>
<td>-0.0643***</td>
<td>1.0152</td>
<td>-0.224***</td>
<td>3.184*</td>
<td>24.17***</td>
</tr>
<tr>
<td>FTSE</td>
<td>-0.0538***</td>
<td>1.0164</td>
<td>-0.275***</td>
<td>3.238**</td>
<td>37.06***</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Series</th>
<th>mean</th>
<th>var.</th>
<th>skewness</th>
<th>kurtosis</th>
<th>JB</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>-0.0584***</td>
<td>1.0121</td>
<td>-0.185***</td>
<td>3.097</td>
<td>15.11***</td>
</tr>
<tr>
<td>DAX</td>
<td>-0.0636***</td>
<td>1.0138</td>
<td>-0.187***</td>
<td>2.983</td>
<td>14.55***</td>
</tr>
<tr>
<td>FTSE</td>
<td>-0.0540***</td>
<td>1.0144</td>
<td>-0.240***</td>
<td>3.070</td>
<td>24.24***</td>
</tr>
</tbody>
</table>

Asterisks *, **, and *** indicate significance at the 10%, 5% and 1% levels, respectively.
Table 6: GARCH(1,1) Value–at–Risk measures, reported is $100 \times \hat{\xi}$

<table>
<thead>
<tr>
<th>Series</th>
<th>$\xi = 0.001$</th>
<th>0.0025</th>
<th>0.005</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>0.36***</td>
<td>0.52**</td>
<td>0.89**</td>
<td>1.69***</td>
<td>3.83***</td>
<td>6.33***</td>
<td>11.01*</td>
</tr>
<tr>
<td>DAX</td>
<td>0.28**</td>
<td>0.65***</td>
<td>1.01***</td>
<td>1.45**</td>
<td>3.79***</td>
<td>6.98***</td>
<td>11.73***</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.60***</td>
<td>0.77***</td>
<td>1.25***</td>
<td>2.02***</td>
<td>3.95***</td>
<td>6.37***</td>
<td>10.56</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Series</th>
<th>0.001</th>
<th>0.0025</th>
<th>0.005</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>0.32***</td>
<td>0.36</td>
<td>0.60</td>
<td>1.17</td>
<td>3.67***</td>
<td>6.25***</td>
<td>11.33***</td>
</tr>
<tr>
<td>DAX</td>
<td>0.20</td>
<td>0.28</td>
<td>0.65</td>
<td>1.13</td>
<td>3.31**</td>
<td>6.98***</td>
<td>12.50***</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.28**</td>
<td>0.65***</td>
<td>0.97***</td>
<td>1.57***</td>
<td>3.67***</td>
<td>6.37***</td>
<td>11.01*</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Series</th>
<th>0.001</th>
<th>0.0025</th>
<th>0.005</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>0.20</td>
<td>0.36</td>
<td>0.56</td>
<td>1.17</td>
<td>3.75***</td>
<td>6.49***</td>
<td>11.45***</td>
</tr>
<tr>
<td>DAX</td>
<td>0.12</td>
<td>0.28</td>
<td>0.65</td>
<td>1.13</td>
<td>3.43***</td>
<td>7.10***</td>
<td>12.54***</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.24*</td>
<td>0.65***</td>
<td>0.85**</td>
<td>1.61***</td>
<td>3.83***</td>
<td>6.57***</td>
<td>11.41**</td>
</tr>
</tbody>
</table>

Asterisks *, **, and *** indicate significance at the 10%, 5% and 1% levels, respectively.
Conditional Skewness

- The results suggest that the innovations, in addition to being leptokurtic, are also skewed, which needs to be taken into account to deliver reliable density forecasts.

- Asymmetric versions of the GED and the \( t \) distributions have been proposed.

- Regarding the GED, the skewed exponential power (SEP) distribution of Fernandez, Osiewalski, and Steel (1995) has density

\[
f(z; p, \theta) = \frac{\theta}{1 + \theta^2 2^{1/p} \Gamma(1/p)} \begin{cases} 
\exp \left\{ -\frac{1}{2} (|z|\theta)^p \right\} & \text{if } z < 0 \\
\exp \left\{ -\frac{1}{2} \left( \frac{z}{\theta} \right)^p \right\} & \text{if } z \geq 0,
\end{cases}
\] (19)

where \( \theta, p > 0 \).

- This distribution nests the normal for \( \theta = 1 \) and \( p = 2 \). For \( \theta < 1(\theta > 1) \), the density is skewed to the left (right), and is fat-tailed for \( p < 2 \).
Skewed Exponential Power Distribution with $p = 1.5$

- $\theta = 1$
- $\theta = 0.75$
- $\theta = 0.5$
• Various skewed versions of the Student’s $t$ exist.

• A $t$ version of (19) is the skewed $t$ distribution proposed by Mittnik and Paolella (2000), which has density

$$f(z; \nu, p, \theta) = \frac{\theta}{1 + \theta^2 \nu^{1/p} B(\nu, 1/p)} \begin{cases} \left(1 + \left(\frac{|z|\theta}{\nu}\right)^p\right)^{-(\nu+1/p)} & \text{if } z < 0 \\ \left(1 + \left(\frac{z}{\theta^p}{\nu}\right)^p\right)^{-(\nu+1/p)} & \text{if } z \geq 0, \end{cases}$$

where $\nu, p, \theta > 0$, and $B(\cdot, \cdot)$ is the beta function.

• In view of our earlier results that the (symmetric) $t$ was somewhat better than the (symmetric) GED, we concentrate on the skewed $t$ distribution (20).
Table 7: GARCH(1,1) estimates for various stock return series, January 1990 to October 2009

<table>
<thead>
<tr>
<th>Series</th>
<th>$\hat{\omega}$</th>
<th>$\hat{\alpha}_1$</th>
<th>$\hat{\beta}_1$</th>
<th>$\hat{\nu}$</th>
<th>$\hat{\theta}$</th>
<th>$\hat{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>0.0302</td>
<td>0.1307</td>
<td>0.9140</td>
<td>4.2942</td>
<td>0.9025</td>
<td>2.1988</td>
</tr>
<tr>
<td></td>
<td>(0.0078)</td>
<td>(0.0142)</td>
<td>(0.0087)</td>
<td>(1.1377)</td>
<td>(0.0151)</td>
<td>(0.1483)</td>
</tr>
<tr>
<td>DAX</td>
<td>0.0237</td>
<td>0.1394</td>
<td>0.9076</td>
<td>3.2897</td>
<td>0.9005</td>
<td>2.2424</td>
</tr>
<tr>
<td></td>
<td>(0.0064)</td>
<td>(0.0153)</td>
<td>(0.0092)</td>
<td>(0.6919)</td>
<td>(0.0143)</td>
<td>(0.1494)</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.0167</td>
<td>0.1431</td>
<td>0.9119</td>
<td>3.8977</td>
<td>0.9100</td>
<td>2.3275</td>
</tr>
<tr>
<td></td>
<td>(0.0043)</td>
<td>(0.0150)</td>
<td>(0.0085)</td>
<td>(1.0776)</td>
<td>(0.0148)</td>
<td>(0.1723)</td>
</tr>
</tbody>
</table>

- All the $\hat{\theta}$s significantly different from 1.
Table 8: Maximized log–likelihood values

<table>
<thead>
<tr>
<th></th>
<th>CAC 40</th>
<th>DAX</th>
<th>FTSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>−8088.5</td>
<td>−8180.9</td>
<td>−6798.8</td>
</tr>
<tr>
<td>Student’s t</td>
<td>−8032.5</td>
<td>−8048.2</td>
<td>−6768.2</td>
</tr>
<tr>
<td>GED</td>
<td>−8048.6</td>
<td>−8085.1</td>
<td>−6779.0</td>
</tr>
<tr>
<td>skewed t</td>
<td>−8013.1</td>
<td>−8024.8</td>
<td>−6749.3</td>
</tr>
</tbody>
</table>

Differences in log–likelihood

<table>
<thead>
<tr>
<th>Difference</th>
<th>CAC 40</th>
<th>DAX</th>
<th>FTSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student’s t – Normal</td>
<td>56.0047</td>
<td>132.6939</td>
<td>30.6056</td>
</tr>
<tr>
<td>GED – Normal</td>
<td>39.8959</td>
<td>95.7972</td>
<td>19.8580</td>
</tr>
<tr>
<td>skew t – t</td>
<td>19.4212</td>
<td>23.3951</td>
<td>18.9364</td>
</tr>
</tbody>
</table>

- The 1% critical value of a $\chi^2(2)$ distribution is 9.2103.
Table 9: GARCH(1,1) density forecasts based on (13)

<table>
<thead>
<tr>
<th>Series</th>
<th>mean</th>
<th>var.</th>
<th>skewness</th>
<th>kurtosis</th>
<th>JB</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>-0.0345*</td>
<td>1.0042</td>
<td>-0.036</td>
<td>3.126</td>
<td>2.163</td>
</tr>
<tr>
<td>DAX</td>
<td>-0.0241</td>
<td>1.0131</td>
<td>-0.029</td>
<td>3.035</td>
<td>0.464</td>
</tr>
<tr>
<td>FTSE</td>
<td>-0.0189</td>
<td>1.0126</td>
<td>-0.088*</td>
<td>3.067</td>
<td>3.630</td>
</tr>
</tbody>
</table>

Asterisks *, **, and *** indicate significance at the 10%, 5% and 1% levels, respectively.
Table 10: GARCH(1,1) Value-at-Risk measures, reported is $100 \times \hat{\xi}$

<table>
<thead>
<tr>
<th>Series</th>
<th>$\xi = 0.001$</th>
<th>0.0025</th>
<th>0.005</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>0.16</td>
<td>0.32</td>
<td>0.40</td>
<td>0.85</td>
<td>2.74</td>
<td>5.56</td>
<td>10.93</td>
</tr>
<tr>
<td>DAX</td>
<td>0.12</td>
<td>0.24</td>
<td>0.40</td>
<td>0.89</td>
<td>2.58</td>
<td>6.25***</td>
<td>11.61***</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.24*</td>
<td>0.24</td>
<td>0.65</td>
<td>1.29</td>
<td>3.15**</td>
<td>5.93**</td>
<td>10.12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Series</th>
<th>0.001</th>
<th>0.0025</th>
<th>0.005</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>0.20</td>
<td>0.36</td>
<td>0.56</td>
<td>1.17</td>
<td>3.75***</td>
<td>6.49***</td>
<td>11.45**</td>
</tr>
<tr>
<td>DAX</td>
<td>0.12</td>
<td>0.28</td>
<td>0.65</td>
<td>1.13</td>
<td>3.43***</td>
<td>7.10***</td>
<td>12.54***</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.24*</td>
<td>0.65***</td>
<td>0.85**</td>
<td>1.61***</td>
<td>3.83***</td>
<td>6.57***</td>
<td>11.41**</td>
</tr>
</tbody>
</table>

Asterisks *, **, and *** indicate significance at the 10%, 5% and 1% levels, respectively.

- Summarizing, both conditional skewness and kurtosis may be important and can considerably improve conditional predictive densities.
Asymmetric GARCH Models

- The basic GARCH model considered so far assumes that the conditional variance $\sigma^2_t$ depends only on the magnitude and not on the sign of past shocks.

- However, stock market variance tends to react more strongly to bad news than to good news, which is often referred to as the leverage effect.

- To illustrate, we may define the leverage effect at lag $\tau$ as

$$L(\tau) = \text{Corr}(\epsilon_{t-\tau}, |\epsilon_t|).$$

(21)
leverage for the CAC 40
leverage for the DAX 40

![Graph showing leverage for the DAX 40 with x-axis from 5 to 50 and y-axis from -0.12 to 0.06. The graph consists of multiple bars representing different data points. The dashed line at y=0 indicates no leverage.]
The first model that has been put forward is the Asymmetric GARCH (AGARCH) of Engle (1990), which specifies the conditional variance as

\[ \sigma_t^2 = \omega + \alpha(\epsilon_{t-1} - \theta)^2 + \beta \sigma_{t-1}^2 \]  \hspace{1cm} (22)

\[ = \omega + \alpha \theta^2 + \alpha \epsilon_{t-1}^2 - 2\alpha \theta \epsilon_{t-1} + \beta \sigma_{t-1}^2. \]  \hspace{1cm} (23)

In model (22), the conditional variance, as a function of \( \epsilon_{t-1} \), has its minimum at \( \theta \) rather than at zero.

Thus, if \( \theta > 0 \), negative shocks will have a greater impact on the conditional variance than positive shocks of the same magnitude.

(23) shows that, if \( \alpha + \beta < 1 \), the unconditional variance of this process is

\[ \text{E}(\sigma_t^2) = \frac{\omega + \alpha \theta^2}{1 - \alpha - \beta}. \]  \hspace{1cm} (24)
Asymmetric GARCH Models II

- The asymmetric GARCH model proposed by Glosten, Jagannathan and Runkle (1993), referred to as GJR–GARCH, models the conditional variance as

\[ \sigma_t^2 = \omega + (\alpha + \theta S_{t-1}) \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \]

where

\[ S_{t-1} = \begin{cases} 
1 & \text{if } \epsilon_{t-1} < 0 \\
0 & \text{if } \epsilon_{t-1} \geq 0 
\end{cases} \]

- Clearly \( \theta > 0 \) implies that the change in the next period’s variance is negatively correlated with today’s return.

- If the innovation density is symmetric (e.g., normal or Student’s t), the unconditional variance is

\[ \text{E}(\sigma_t^2) = \frac{\omega}{1 - \alpha - \theta/2 - \beta}. \]
News Impact Curve

- To analyze the asymmetric response of the variance in different GARCH specifications, Engle and Ng (1993) defined the new impact curve (NIC).

- This is defined as the functional relationship

\[ \sigma_t^2 = \sigma_t^2(\epsilon_{t-1}), \]

with all lagged variances evaluated at their unconditional values.

- For example, for the standard symmetric GARCH(1,1) model, we have

\[ \sigma_t^2(\epsilon_{t-1}) = A + \alpha \epsilon_{t-1}^2, \]

where

\[ A = \omega + \beta \sigma^2, \quad \sigma^2 = \frac{\omega}{1 - \alpha - \beta}. \]

- This is a symmetric function of \( \epsilon_{t-1} \).
• Asymmetries may be introduced in various ways: Compared to the standard GARCH, we can change either the position of the slope of the NIC (or both).

• For example, the AGARCH captures asymmetry by allowing its NIC to be centered at a positive $\epsilon_{t-1}$ (position), since

$$\sigma_t^2(\epsilon_{t-1}) = A + \alpha(\epsilon_{t-1} - \theta)^2.$$  

• The GJR captures the asymmetry in the impact of news on volatility via a steeper slope for negative than for positive shocks, i.e.,

$$\sigma_t^2(\epsilon_{t-1}) = A + \begin{cases} (\alpha + \theta)\epsilon_{t-1}^2 & \text{if } \epsilon_{t-1} < 0 \\ \alpha\epsilon_{t-1}^2 & \text{if } \epsilon_{t-1} \geq 0, \end{cases}$$

but the NIC of the GJR is still centered at zero, i.e., $\sigma_t^2(\epsilon_{t-1})$ is minimized for $\epsilon_{t-1} = 0$. 


Table 11: Asymmetric GARCH(1,1) estimates for various stock return series, January 1990 to October 2009 (S&P 500: March 2010)

<table>
<thead>
<tr>
<th>Series</th>
<th>( \hat{\omega} )</th>
<th>( \hat{\alpha}_1 )</th>
<th>( \hat{\beta}_1 )</th>
<th>( \hat{\theta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>0.0000 (0.0073)</td>
<td>0.0621 (0.0069)</td>
<td>0.9187 (0.0084)</td>
<td>0.7361 (0.0954)</td>
</tr>
<tr>
<td>DAX</td>
<td>0.0087 (0.0069)</td>
<td>0.0709 (0.0073)</td>
<td>0.9081 (0.0088)</td>
<td>0.6524 (0.0829)</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.0000 (0.0036)</td>
<td>0.0673 (0.0071)</td>
<td>0.9189 (0.0079)</td>
<td>0.4693 (0.0664)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Series</th>
<th>( \hat{\omega} )</th>
<th>( \hat{\alpha}_1 )</th>
<th>( \hat{\beta}_1 )</th>
<th>( \hat{\theta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAC 40</td>
<td>0.0297 (0.0050)</td>
<td>0.0157 (0.0067)</td>
<td>0.9184 (0.0086)</td>
<td>0.0959 (0.0109)</td>
</tr>
<tr>
<td>DAX</td>
<td>0.0364 (0.0053)</td>
<td>0.0220 (0.0072)</td>
<td>0.9042 (0.0093)</td>
<td>0.1049 (0.0126)</td>
</tr>
<tr>
<td>FTSE</td>
<td>0.0119 (0.0021)</td>
<td>0.0187 (0.0064)</td>
<td>0.9227 (0.0073)</td>
<td>0.0943 (0.0104)</td>
</tr>
</tbody>
</table>
Table 12: Maximized log–likelihood values

<table>
<thead>
<tr>
<th>Model</th>
<th>CAC 40</th>
<th>DAX</th>
<th>FTSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>GARCH</td>
<td>-8088.5</td>
<td>-8180.9</td>
<td>-6798.8</td>
</tr>
<tr>
<td>AGARCH</td>
<td>-8045.0</td>
<td>-8141.8</td>
<td>-6761.2</td>
</tr>
<tr>
<td>GJR–GARCH</td>
<td>-8043.8</td>
<td>-8138.5</td>
<td>-6755.2</td>
</tr>
</tbody>
</table>

Differences in log–likelihood

<table>
<thead>
<tr>
<th></th>
<th>CAC 40</th>
<th>DAX</th>
<th>FTSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>AGARCH – GARCH</td>
<td>43.5299</td>
<td>39.0356</td>
<td>37.6334</td>
</tr>
<tr>
<td>GJR – GARCH</td>
<td>44.6940</td>
<td>42.3483</td>
<td>43.6754</td>
</tr>
</tbody>
</table>
NICs for CAC 40

\[ \sigma_t^2 \]

\( \varepsilon_{t-1} \)

GARCH
AGARCH
GJR-GARCH
NICs for DAX 30

\[ \sigma_t^2 \]

\( \varepsilon_{t-1} \)

GARCH
AGARCH
GJR-GARCH