Problem Set 5

Problem 1 Consider the following model for the returns, $r_t$, of a market index, $r_t$.

\[
\begin{align*}
    r_t &= \mu + \delta \sigma_t + \epsilon_t \\
    \epsilon_t &= \eta_t \sigma_t, \quad \eta_t \sim \text{iid } N(0, 1) \\
    \sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2.
\end{align*}
\]

Is this an economically plausible specification? What is the expected sign of parameter $\delta$? What would you expect $\mu$ to be (consider what happens if $\sigma_t^2 = 0$)?

Problem 2 Consider the GARCH(1,1) process

\[
\begin{align*}
    \epsilon_t &= \eta_t \sigma_t, \quad \eta_t \sim \text{iid } N(0, 1) \\
    \sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2.
\end{align*}
\]

We want to calculate variance forecasts. That is, for $\tau \geq 1$, find

\[
\text{Var}(\epsilon_{t+\tau} | I_t) = \mathbb{E}(\epsilon_{t+\tau}^2 | I_t),
\]

where $I_t = \{\epsilon_s : s \leq t\}$ is the information set at time $t$. If we have a model for daily returns and we are interested in, for example, weekly or monthly returns, we also need quantities of the form

\[
\text{Var}(\epsilon_{t+1} + \cdots + \epsilon_{t+\tau} | I_t).
\]

Problem 3 Suppose that the return $r_t$ of your portfolio is generated by

\[
\begin{align*}
    r_t &= 0.025 + \epsilon_t \\
    \epsilon_t &= \eta_t \sigma_t, \quad \eta_t \sim \text{iid } N(0, 1) \\
    \sigma_t^2 &= 0.025 + 0.075 \epsilon_{t-1}^2 + 0.9 \sigma_{t-1}^2.
\end{align*}
\]

Your current estimate for $\sigma_t^2$ is its unconditional expectation. Unfortunately, however, due to unpredictable adverse market conditions, your portfolio suffers from an unusually large negative shock, so that $r_t = -4.75$. Calculate the 1% Value–at–Risk for period $t + 1$. 1
**Problem 4** Consider the model for a stock return, $r_t$,

\begin{align*}
  r_t &= \mu + \epsilon_t \\
  \epsilon_t &= \eta_t \sigma_t, \quad \eta_t \overset{iid}{\sim} N(0,1) \\
  \sigma_t^2 &= \omega + \alpha (\epsilon_{t-1} - \theta)^2 + \beta \sigma_{t-1}^2.
\end{align*}

Find $\text{Cov}(\epsilon_{t-1}, \epsilon_t^2)$ and explain.

**Problem 5** Consider two different processes of the form considered in Problem 4. Both processes have the same unconditional variance and common parameters $\alpha$ and $\beta$. However, the first has $\theta = 0$, whereas $\theta > 0$ for the second. Show that the second process has a greater kurtosis than the first, so that, at least to some extent, accounting for the leverage effect also explains (at least a moderate) part of the excess kurtosis that plagues symmetric Gaussian GARCH processes.