Problem 1

Suppose that the Single–Index Model (SIM) describes the percentage returns of individual stocks, \( r_i, i = 1, \ldots, N \), that is, with \( r_M \) being the return of the market,

\[
    r_i = \alpha_i + \beta_i r_M + \epsilon_i, \quad E(\epsilon_i) = 0, \quad \text{Var}(\epsilon_i) = \sigma_i^2, \quad i = 1, \ldots, N;
\]

\[
    \text{Cov}(\epsilon_i, \epsilon_j) = 0, \quad i \neq j; \quad \text{Cov}(\epsilon_i, r_M) = 0, \quad i = 1, \ldots, N;
\]

\[
    E(r_M) = \mu_M, \quad \text{Var}(r_M) = \sigma_M^2.
\]

(a) Find \( \text{Var}(r_i), i = 1, \ldots, N, \) and \( \text{Corr}(r_i, r_j) \), the correlation between \( r_i \) and \( r_j \).

(b) Show that, if \( \beta_i \) and \( \beta_j \) have the same sign, the correlation between \( r_i \) and \( r_j \) is increasing in the variance of the market return, \( \sigma_M^2 \).

Problem 2

Your portfolio consists of two risky assets, the percentage returns of which are bivariate normally distributed with mean vector, \( \mu \), and covariance matrix, \( \Sigma \), given by

\[
    \mu = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix},
\]

respectively. The weight vector of your portfolio is \( w = [0.5, 0.5]' \), and you have invested 5000 $ in this portfolio. Compute the 1% Value–at–Risk for this position.
Problem 3

Assume that the return of your portfolio, $r_p$, is normally distributed with mean $\mu$ and variance $\sigma^2$, i.e., its density function is given by

$$f_{\text{Normal}}(r_p; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{ -\frac{(r_p - \mu)^2}{2\sigma^2} \right\}, \quad r_p \in \mathbb{R}. \quad (1)$$

(a) Assume that $\mu = 2.5$ and $\sigma^2 = 2.25$. Your portfolio is worth 1 $. Compute the 1% Value–at–Risk.

(b) Now assume that the portfolio return has a Laplace (or double exponential) distribution with mean $\mu$ and variance $\sigma^2$, i.e., its density function is

$$f_{\text{Laplace}}(r_p; \mu, \sigma^2) = \frac{1}{\sqrt{2\sigma}} \exp\left\{ -\sqrt{2} \left| \frac{r_p - \mu}{\sigma} \right| \right\}, \quad r_p \in \mathbb{R}. \quad (2)$$

The Laplace distribution has sometimes been found to be useful for modeling stock return distributions, see also Figure 1 for the standardized densities with $\mu = 0$ and $\sigma^2 = 1$.

(i) Find the kurtosis of the Laplace distribution given in (2).

(ii) Assume that, in (2), $\mu = 2.5$ and $\sigma^2 = 2.25$. Find the 1% Value–at–Risk for a portfolio which is worth 1 $. Compare the result with what has been obtained in part (a).

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Figure 1: Standard (i.e., zero mean and unit variance) normal (dashed) and Laplace (solid) densities, given by (1) and (2), respectively, with $\mu = 0$ and $\sigma^2 = 1$. 