NOTES. TIME SERIES ANALYSIS

February 8: Part One

Motivation and the First Order Deterministic Case
A. Why this extended study of difference equations?
   a. Put simply, to understand deep implication of writing more exact equations.

1. Recall point about conceptualization vs. modeling.
   a. Conceptualize social processes in terms of trends, cycles, and "irregular" points-shocks.

T.S.: Decomposition

Eq. talk about trends towards peace or cycle of violence.

But when we go to model processes we write down equations.
FIGURE 1.1 Hypothetical Time Series
and momentary surges and declines. Overall, robberies appear to be a mix of an increasing trend, at least one cycle, and a series of shocks.

Figure 1: Robberies and Unemployment in the U.S.

While we conceptualize social processes in terms of time functions, we model them as equations. As explained in Chapter One, because of the way we measure our variables, these equations usually are expressed in discrete rather than continuous time.\(^2\) Illustrative of the univariate equations we use are

\[ R_t = \alpha R_{t-1} + \epsilon_t \]  \hspace{1cm} (1)

and

\[ R_t = \alpha_0 + \alpha_1 R_{t-1} + \alpha_2 R_{t-2} + \beta t + \epsilon_t \]  \hspace{1cm} (2)

\(^2\)We think that processes like international conflict and cooperation, crime, and popular evaluation of the President are continuously ongoing. But we only (systematically) sample these processes at discrete times and almost always a regular intervals. Continuous time models are used occasionally. In this case, scholars model social processes in terms of deterministic or stochastic differential equations. Gillespie et al (1977), Schrodt (1979), Freeman and Duvall (1982), Freeman (1983), Zinnes and Muncaster (XXX), and Mebane (2000) are illustrative. But, again, these are exceptions. As regards the translation from verbal concepts to mathematical equations, graph algebra can be used for this purpose. Cortés, Przeworski and Sprague (1974) explain this algebra and show how it can be applied to the study of conflict dynamics and other social problems. This algebra is not widely used in the social sciences, however.
4. Goal: understand how/extent once specified and estimate output

(a) Make of Trends, cyclic, steady, etc., RESPONSE SERIES

(b) ENDS

(c) SUBMIT ASSUMPTIONS FOR ESTIMATES - Jan Next Week

(d) ALTERNATIVE/New Ideas about TENDENCY TODAY

⇒ 70 1/2°
2. Locating our study of difference equations in the calculus of finite differences - The "L" word

a. A look at some equations

- Univariate

- Univariate

- Univariate

Tack Through -

Explain →
\[ R_t = \alpha R_{t-1} + \epsilon_t \]  
\[ R_t = \alpha_0 + \alpha_1 R_{t-1} + \alpha_2 R_{t-2} + \beta t + \epsilon_t \]  
\[ R_t = tR_{t-1} + \epsilon_t \]  
\[ R_{t+2} = (t + 2)R_{t+1} - tR_t + t + \epsilon_t \]  
\[ R_t = \alpha_0 (R_{t-1})^2 + \alpha_1 R_{t-2} + \epsilon_t \]  
\[ R_{t+1} = -tR_{t+1}R_t + R_t + \epsilon_t \]
5. Note distinction

i. Homogeneous - Nonhomogeneous

- If function of itself only

\[ R_x = \theta R_{x+1} \]

\[ R_x - \theta R_{x-1} = 0 \]

- Nonhomogeneous: determinate - undetermined

ii. Order

- For which variable

- Points of past related to present

iii. Linear - nonlinear

Form of field:

Put together (3) form, nonhomogeneous, linear, with variable coefficient:

(3): second order, nonhomogeneous, also related with constant coefficient
Consider:

(1.1) \[ y_{t+1} + 2y_t = 0 \]

(1.2) \[ y_{t+2} - y_t = 0 \]

(1.3) \[ y_{t+2} + 2y_{t+1} + (1-t)y_t = 2t + 5 \]

(1.4) \[ y_t y_{t+3} - 3y_t y_{t+1} - y_t^2 = 2 \]

(1.5) \[ (y_{t+1} - y_t)^2 + y_t^2 = -1 \]

(1.6) \[ y_{t+1} - 3y_{t+1} y_t + 2y_t^2 = 0 \]
The set $S$ defined over all values of $t$ is each function of $t$ (but not of $w$).

Where $f_0, f_1, \ldots, f_n$ and $9$ are

$$f_0(t) + f_1(t) y_0 + \cdots + f_n(t) y_n = g(t)$$

In the form:

is linear if it can be written

A difference can be a set $S$
3. Multivariate case ⇒ System of Equations (Even if we only write one, several are implied)

a. Revisit R = Unemployment

b. Equations ⇒ System R, U \{ THE SIMPLE ONE \}

- 10 → 12, explain exogeneity
- But also linear, so nonlinear still
  \( \ldots \)
  \( \ldots \)

- An for UC, EU system

4. Table 1. (Without DET (STRETCH))

- Only linear, constant \( (I, III) \)
  a. Unregual - old regression
  b. Pedagogy
  c. Base to con \( \mathbf{X} + \mathbf{Y} \) ⇒ Multiple Equation

- Extra Credit Problem
\[ R_t = \alpha_0 + \alpha_1 R_{t-1} + \beta_1 U_{t-1} + \epsilon_t \]  

(10)

\[ R_t = \alpha_0 + \alpha_1 R_{t-1} + \beta_1 U_{t-1} + \epsilon_{1t} \]

\[ U_t = \gamma_0 + \gamma_3 U_{t-3} + \epsilon_{2t} \]  

(12)

\[ UC_t = \alpha_0 + \alpha_1 UC_{t-1} + \alpha_2 CU_{t-1} + \epsilon_{1t} \]  

(13)

\[ CU_t = \beta_0 + \beta_1 CU_{t-1} + \beta_2 UC_{t-1} + \epsilon_{2t} \]

\[ UC_{t+1} = \alpha_0 t UC_t + \alpha_1 CU_t + \epsilon_{1t} \]  

(14)

\[ CU_{t+1} = \beta (UC_t)^2 + \epsilon_{2t} \]
Table 1: Some Types of Difference Equation Models

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Linear</th>
<th>Nonlinear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>I</td>
<td>IV</td>
</tr>
<tr>
<td>Time Varying</td>
<td>II</td>
<td>V</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Multivariate</th>
<th>Constant</th>
<th>Time Varying</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Multiple equation)</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 2.2 (TSSS Chp. 2 Mm. 13.15)

Claim: \( y_t = 0.33 \times (-2)^{t-0} \)

Step 1. Verify initial condition satisfied.

\[
y_0 = 0.33 - 33(-2) = 33 - 33 = 0
\]
Step 2. Confirm that the function makes equation true for all $t > 0$.

Recall $y^* = -2y_{t-1} + 1$

Claim $y^*_t = (3.3 - 3.3(-2))^{t-1} + 1$

Implicating $y^*_{t-1} = -6.6 + 6.6(-2)^{t-2} + 1$

$= 3.3 - 3.3(-2)^t$

Sequence: 30, 99, 297, -495, 1089, ...
The Linear First Order Difference Eqn.

$y_{t+1} = Ay_t + B$

$y_0 = C$

Of the form

$f(t)$ $y_{t+1} \overset{f}{\rightarrow} f,c(t)$ $y_0 = g(t)$

$\cdots$

$\Rightarrow t=0$

$g(0)$ $y_t = \frac{CA^t + B}{1-A}$ if $A \neq 1$

$y_t =$ General Solution

$\begin{cases} C + Bt & \text{if } A = 1 \\ \frac{CA^t + B}{1-A} & \text{if } A \neq 1 \end{cases}$
\[
y_t = C A^t + B \left( \frac{1-A^t}{1-A} \right) \quad A \neq 1
\]

Example - From TSASS, Example 2.3 - WE JUST DID THIS ONE!

\[
y_t = -2y_{t-1} + 1 \\
y_0 = 0
\]

So

\[
A = -2, B = 1, C = 0, A \neq 1
\]

\[
y_t = 0 \cdot (-2)^t + 1 \cdot \left( \frac{1-(-2)^t}{1+2} \right)
\]

\[
= \frac{1}{3} \left( 1 - (-2)^t \right) = 0.33 - 0.33(-2)^t
\]
\[ y_{t+1} = A y_t + B \]

\[ y_t = C \left( A^t + B \right) \frac{1-A^t}{1-A} \]

**Examples:**

**EQN.** \( y_{t+1} = 2y_t + 1 \)

**SOLN.** \( y_t = 5 \cdot 2^t + 1 \cdot \frac{1-2^t}{1-2} \)

\[ y_t = 6 \cdot 2^t - 1 \]

**EQN.** \( 2y_{t+1} - y_t = 4 \)

**STANDARD FORM SOLN.**

\( y_{t+1} = \frac{1}{2} y_t + 2 \)

\[ y_t = \left( \frac{1}{2} \right)^t \cdot 3 + 2 \cdot \frac{1-\left( \frac{1}{2} \right)^t}{1-\frac{1}{2}} \]

\[ y_t = 4 - \left( \frac{1}{2} \right)^t \]
THE EQUATION \( y_{k+1} = Ay_k + B \)

**Theorem 2.3.** The linear first-order difference equation

\[
y_{k+1} = Ay_k + B \quad k = a, a + 1, a + 2, \cdots
\]

taken over the indicated set of \( k \)-values (which may or may not continue indefinitely) has infinitely many solutions. If \( y \) is a solution, there is a constant \( C \) such that

\[
y_k = \begin{cases} 
CA^{k-a} + B \frac{1 - A^{k-a}}{1 - A} & \text{if } A \neq 1 \\
C + B(k - a) & \text{if } A = 1.
\end{cases}
\]

If a single value of \( y \) is prescribed for one of the \( k \)-values \( a, a + 1, a + 2, \cdots \), then a unique solution of (2.39) is determined. In particular, if \( y_a \) is prescribed, then the unique solution of (2.39) is given by (2.40) with \( C = y_a \).

Note that when \( a = 0 \), these results reduce to those of Theorem 2.2 and its corollary.
\[ Y_t = A y_{t-1} + B \]
\[ y_{t+1} = A y_t + B \]

**Soln. For Case A \neq 1.**

\[ y_t = A^t y_0 + B \left( \frac{1 - A^t}{1 - A} \right) \quad k = 0, 1, 2, \ldots \]

This can be written

\[ y_t = A^t y_0 + \frac{B}{1 - A} - \frac{B}{1 - A} A^t \]

\[ = A^t (y_0 - \frac{B}{1 - A}) + \frac{B}{1 - A} \]

\[ y_t - \frac{B}{1 - A} = A^t (y_0 - \frac{B}{1 - A}) \]

\[ y_t - y^* = A^t (y_0 - y^*) \]

Let \( y^* = \frac{B}{1 - A} \)

---

"Continue"
Dynamic Impedance

Supply for case, along with

\[
\text{Table 2 in TASS (4.28)}
\]

Where

\[
y^* = \frac{1-a}{\delta}
\]

\[
y^* = A^* (y^* - y^*)^* + y^*
\]

\[
y^* - y^* = A^* (y^* - y^*)
\]
\[ y_{k+1} = Ay_k + B \quad k = 0, 1, 2, \ldots \]

<table>
<thead>
<tr>
<th>Hypotheses</th>
<th>Conclusions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A \neq 1)</td>
<td>(y_0 = y^<em>) \hspace{1cm} (y_k = y^</em>) \hspace{1cm} constant (=y*)</td>
</tr>
<tr>
<td>(A &gt; 1)</td>
<td>(y_0 &gt; y^<em>) \hspace{1cm} (y_k &gt; y^</em>) \hspace{1cm} monotone increasing, diverges to +\infty</td>
</tr>
<tr>
<td>(A &gt; 1)</td>
<td>(y_0 &lt; y^<em>) \hspace{1cm} (y_k &lt; y^</em>) \hspace{1cm} monotone decreasing, diverges to -\infty</td>
</tr>
<tr>
<td>(0 &lt; A &lt; 1)</td>
<td>(y_0 &gt; y^<em>) \hspace{1cm} (y_k &gt; y^</em>) \hspace{1cm} monotone decreasing, converges to limit y*</td>
</tr>
<tr>
<td>(0 &lt; A &lt; 1)</td>
<td>(y_0 &lt; y^<em>) \hspace{1cm} (y_k &lt; y^</em>) \hspace{1cm} monotone increasing, converges to limit y*</td>
</tr>
<tr>
<td>(-1 &lt; A &lt; 0)</td>
<td>(y_0 \neq y^<em>) \hspace{1cm} damped oscillatory, converges to limit y</em></td>
</tr>
<tr>
<td>(A = -1)</td>
<td>(y_0 \neq y^*) \hspace{1cm} divergent, oscillates finitely</td>
</tr>
<tr>
<td>(A &lt; -1)</td>
<td>(y_0 \neq y^*) \hspace{1cm} divergent, oscillates infinitely</td>
</tr>
<tr>
<td>(A = 1)</td>
<td>(B = 0) \hspace{1cm} (y_k = y_0) \hspace{1cm} constant (=y_0)</td>
</tr>
<tr>
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The various behavior types listed in Table 2.2 are roughly sketched in Figure 2.3. In each case, the graph is selected to be typical of the type of solution obtained for the specified values of \(A\), \(B\), and \(y_0\). The graphs are labeled to match the rows of Table 2.2.

In the sections to follow we shall apply these results to a wide variety of difference equations which arise in the social and behavioral sciences. For purposes of easy reference, we summarize our findings in the following theorem.
Figure 2.3