Notes. Time Series Analysis

February 1, Part Two

More on Difference Equations
Figure 1.2 Convergent and nonconvergent sequences.

- **(a)** $y_t = 0.9y_{t-1} + \epsilon_t$
- **(b)** $y_t = 0.5y_{t-1} + \epsilon_t$
- **(c)** $y_t = -0.5y_{t-1} + \epsilon_t$
- **(d)** $y_t = y_{t-1} + \epsilon_t$
- **(e)** $y_t = 1.2y_{t-1} + \epsilon_t$
- **(f)** $y_t = -1.2y_{t-1} + \epsilon_t$
Consider the equation

\[ y_t = 2y_{t-1} + 1, \quad y_0 = 5 \]

Claim: Solution is

\[ y_t = 6^t - 1. \]

Check: For \( t = 0 \)

\[ y_0 = 6^0 - 1 = 6 - 1 = 5. \]
Recall Eqn. is

\[ y_t = 2y_{t-1} + 1 \]

for all other \( t \)

\[ 6.2^{t-1} - 1 \]

\[ = \frac{[6.2^{t-1} - 1]}{2} + 1 \]

\[ 6.2^{t} - 1 = 6.2^{t-1} - 2 + 1 \]

\[ \frac{1}{1.05} \]

Solution is also \( 35, 1123, 4795, \ldots \)
Another Example

\[ y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \epsilon_t \]

**Second Order Difference Eqn.**

\[ y_t = 3y_{t-1} - 2y_{t-2} - 1 \]

\[ y_0 = -3 \quad y_1 = 5 \]

Claim: Solution is:

\[ y_t = -10 + 7.2^t + t \]

NB: JT
\[ y_t = 3y_{t-1} - 2y_{t-2} - 1 \]

**Check solution for \( t = 0 \).**

Recall \( y_0 = -3 \), \( y_1 = 5 \). 

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-10 + 7.2 + 0)</td>
</tr>
<tr>
<td>1</td>
<td>(-10 + 7.2 + 1)</td>
</tr>
<tr>
<td>2</td>
<td>(-10 + 7.2 + 1)</td>
</tr>
<tr>
<td>3</td>
<td>(-10 + 7.2 + 1)</td>
</tr>
<tr>
<td>4</td>
<td>(-10 + 7.2 + 1)</td>
</tr>
</tbody>
</table>

For \( t = 1 \):

\[ y_1 = -10 + 7.2 + 1 = -1 + 14 + 1 = 5 \]
For $t \geq 1$

$$T_{2.1} = -2(10 + 7.2 + t - 2) - 1$$

$$T_{3.1} = 3(10 + 7.2 + t - 1)$$

$$-10 + 7.2 + t = 3(10 + 7.2 + t - 1)$$

$$= -30 + 21.2 + 3t - 3 + 20$$

$$= -10 + 21.2 + t - 14.2 + t - 4 - 1$$

$$= -10 + 14.2 + t + t$$

$$= 7 + 7.2 + t$$

Sequence: $3, 5, 7, 9, \ldots$
**Third Example** (See Goldberg, p. 136)

\[ A(r) = -3y_{t-1} - y_{t-2} \]

\[ r^2 + 3r - 1 = 0 \]

\[ r = -\frac{3}{2} \pm \frac{\sqrt{9-4}}{2} \]

\[ y_t + 3y_{t-1} + y_{t-2} = 0 \]

\[ y_t = C_1 \left( \frac{-3 + \sqrt{5}}{2} \right)^t + C_2 \left( \frac{-3 - \sqrt{5}}{2} \right)^t \]

Sequence depends on constants, but values clearly will oscillate.
ANOTHER EXAMPLE

\[ y_t = \frac{1}{10} y_{t-1} \quad \checkmark \]

OR,

\[ y_t - \frac{1}{10} y_{t-1} = 0. \]

CLAIM: SOLUTION IS "OF FORM"

\[ y_t = C \cdot \left( \frac{1}{10} \right)^t \]

\[ y_0 = C \]

WHERE C IS A CONSTANT.
\( y_{t-1} - \frac{1}{10} y_{t-1} = 0 \quad \forall t. \)

For we have

\[
\begin{align*}
C(t) + (\frac{1}{10})^t (c(t) + \frac{1}{10}) &= 0 \\
(\frac{1}{10})^t (c(t) + (\frac{1}{10})^t = 0 \\
(\frac{1}{10})^t (c(t) - c(t) + \frac{1}{10}) &= 0 \\
1 &= 0
\end{align*}
\]

So with \( C \) and assuming many values.

So we have many solutions to our equation.
Solution:

\[ y_e = c \left( \frac{1}{10} \right)^t \]

But say

\[ y_0 = 0. \quad \text{Then } c = 0 \quad \text{so } 0 \cdot \left( \frac{1}{10} \right)^0 = 0 \]

\[ y_0 = 1. \quad \text{Then } c = 1 \quad \text{so } 1 \cdot \left( \frac{1}{10} \right)^0 = 1 \]

\[ y_0 = 100. \quad \text{Then } c = 100 \quad \text{so } 100 \cdot \left( \frac{1}{10} \right)^0 = 100 \]

These are so-called "particular solutions," related to initial condition or value of \( y_t \) when \( t = 0 \).
THEOREM 2.1 (GOLDBERG, p. 61)

THE LINEAR DIFFERENCE EQUATION OF ORDER N

\[ f_0(t)y_{t+n} + f_1(t)y_{t+n-1} + \ldots + f_n(t)y_t = g(t) \]

over a set S of consecutive integer values of t has one, and only one, solution y for which values at N consecutive t-values are arbitrarily prescribed.
Solutions - Our time functions are also sequences for \( t = 0, 1, 2, \ldots \)

\[
\left( \frac{1}{10} \right)^t = \{1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots \}
\]

\[
(-2)^t = \{1, -2, 4, -8, 16, \ldots \}
\]

\[
\left\{ \frac{1}{t+1} \right\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \}
\]

\[
(1 + (-1)^t) = \{2, 0, 2, 0, 2, \ldots \}
\]
SEQUENCES

Table 2.1. (These are not the only possibilities, but they suffice for our purposes.)

**Table 2.1**

**Convergent Sequences**

1. Constant
2. Bounded and monotone increasing
3. Bounded and monotone decreasing
4. Damped oscillatory

**Divergent Sequences**

D1. Diverges to $+\infty$
D2. Diverges to $-\infty$
D3. Oscillates finitely
D4. Oscillates infinitely

Some examples of each of the behaviors given in Table 2.1 are:

**C1. Constant:**

| $\{2\}$ | 2, 2, 2, 2, 2, · · · | (limit = 2) |

**C2. Bounded, monotone increasing:**

| $\left\{ \frac{3^k - 1}{3^k} \right\}$ | 0, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{4}{3}$, $\frac{8}{3}$, · · · | (limit = 1) |

| $\left\{ 5 - \frac{1}{2^k} \right\}$ | 4, 4$\frac{1}{2}$, 4$\frac{3}{4}$, 4$\frac{7}{8}$, 4$\frac{15}{16}$, · · · | (limit = 5) |

| $\left\{ \frac{2k}{k + 1} \right\}$ | 0, 1, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, · · · | (limit = 2) |

**C3. Bounded, monotone decreasing:**

| $\left\{ \frac{500}{k + 1} \right\}$ | $\frac{500}{1}$, $\frac{500}{2}$, $\frac{500}{3}$, $\frac{500}{4}$, $\frac{500}{5}$, · · · | (limit = 0) |

| $\left\{ 5 + \frac{1}{2^k} \right\}$ | 6, 5$\frac{1}{2}$, 5$\frac{3}{4}$, 5$\frac{7}{8}$, 5$\frac{15}{16}$, · · · | (limit = 5) |
C4. Damped oscillatory:
\[
\{1 + (-\frac{1}{2})^k\} \quad 2, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 1, \frac{1}{2}, \cdots
\]
\[
\left\{\frac{1000}{(-5)^k}\right\} \quad 1000, -200, 40, -8, 1\frac{1}{2}, \cdots
\]

D1. Diverges to \(+\infty\):
\[
\{3^k\} \quad 1, 3, 9, 27, 81, \cdots
\]
\[
\{1.01^k\} \quad 1, 1.01, 1.0201, 1.030301, \cdots
\]
\[
\{k\} \quad 0, 1, 2, 3, 4, \cdots
\]
\[
\{k + (-1)^k\} \quad 1, 0, 3, 2, 5, \cdots
\]

D2. Diverges to \(-\infty\):
\[
\{1 - 2^k\} \quad 0, -1, -3, -7, -15, \cdots
\]
\[
\{1 - 4k\} \quad 1, -3, -7, -11, -15, \cdots
\]

D3. Oscillates finitely:
\[
\{(-1)^k\} \quad 1, -1, 1, -1, 1, \cdots
\]
\[
\left\{5 + (-1)^k \frac{k}{k + 1}\right\} \quad 5, 4\frac{1}{2}, 5\frac{5}{8}, 4\frac{1}{4}, 5\frac{1}{2}, \cdots
\]

D4. Oscillates infinitely:
\[
\{1 + (-2)^k\} \quad 2, -1, 5, -7, 17, \cdots
\]
\[
\{(-1)^k k\} \quad 0, -1, 2, -3, 4, \cdots
\]
The Linear First Order Difference Eqn.

\[ Y_{t+1} = A Y_t + B \]

of the form

\[ Y_{t+1} - A Y_t = B \]

General solution

\[ Y_t = \begin{cases} C A^t + B \frac{1 - A^t}{1 - A} & \text{if } A \neq 1 \\ C + Bt & \text{if } A = 1 \end{cases} \]

\[ Y_{t+1} = A Y_t + B \]

\[ Y_{t+1} = Y_t + B \]
\[ y_{t+1} = Ay_t + B \]

\[ y_t = CA^t + B \frac{1-A^t}{1-A} \quad A \neq 1 \]

**Examples.**

<table>
<thead>
<tr>
<th>Eqn.</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_{t+1} = 2y_t + 1 ) &amp; ( y_0 = 5 )</td>
<td>( y_t = 5 \cdot 2^t + 1 \cdot \frac{1-2^t}{1-2} )</td>
</tr>
<tr>
<td>( y_{t+1} = 6 \cdot 2^t - 1 ) &amp; ( y_0 = 5 )</td>
<td>( y_t = 6 \cdot 2^t - 1 )</td>
</tr>
</tbody>
</table>

**Note:** \( A \neq 1 \)

**A = 1**

<table>
<thead>
<tr>
<th>Eqn.</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2y_{t+1} - y_t = 4 ) &amp; ( y_0 = 3 )</td>
<td>( y_{t+1} = \frac{1}{2} y_t + 2 )</td>
</tr>
<tr>
<td>( y_t = (\frac{1}{2})^t \cdot 3 + 2 \cdot \frac{1-\frac{1}{2}^t}{1-\frac{1}{2}} )</td>
<td>( y_t = 4 - (\frac{1}{2})^t )</td>
</tr>
</tbody>
</table>

**Standard Form**

\[ 3 \left( \frac{1}{2} \right)^t + 2 \cdot \left( \frac{1-\frac{1}{2}^t}{\frac{1}{2}} \right) \]

\[ 3 \left( \frac{1}{2} \right)^t + 4 \left( 1 - \frac{1}{2}^t \right) \]

\[ 3 \left( \frac{1}{2} \right)^t - 4 \left( \frac{1}{2} \right)^t + y \]

\[ y - (\frac{1}{2})^t \]
THE EQUATION \( y_{k+1} = Ay_k + B \)

**Theorem 2.3.** The linear first-order difference equation

\[(2.39) \quad y_{k+1} = Ay_k + B \quad k = a, a + 1, a + 2, \ldots\]

taken over the indicated set of \( k \)-values (which may or may not continue indefinitely) has infinitely many solutions. If \( y \) is a solution, there is a constant \( C \) such that

\[(2.40) \quad y_k = \begin{cases} CA^{k-a} + B \frac{1 - A^{k-a}}{1 - A} & \text{if } A \neq 1 \\ C + B(k - a) & \text{if } A = 1. \end{cases} \quad k = a, a + 1, a + 2, \ldots\]

If a single value of \( y \) is prescribed for one of the \( k \)-values \( a, a + 1, a + 2, \ldots \), then a unique solution of (2.39) is determined. In particular, if \( y_a \) is prescribed, then the unique solution of (2.39) is given by (2.40) with \( C = y_a \).

Note that when \( a = 0 \), these results reduce to those of Theorem 2.2 and its corollary.
**CASE A=1**

**Solution for Case A=1:**

\[
Y_t = A^t y_0 + \frac{B}{1-A} - \frac{B}{1-A}\left(\frac{B}{1-A}\right)^t
\]

\[
Y_{t+1} = A^t y_0 + B + \frac{B}{1-A} - \frac{B}{1-A}\left(\frac{B}{1-A}\right)^{t+1}
\]

This can be written as:

\[
Y_t = A^t y_0 + \frac{B}{1-A} - \frac{B}{1-A}\left(\frac{B}{1-A}\right)^{t-1}
\]

\[
y_t - \frac{B}{1-A} = A^t y_0 - \frac{B}{1-A}
\]

\[
y_{t+1} - \frac{B}{1-A} = A^t\left(\frac{y_0 - \frac{B}{1-A}}{1-A}\right)
\]

\[
y_t - \frac{B}{1-A} = A^t\left(y_0 - \frac{B}{1-A}\right)
\]

Continue...
\[ y_t - y^* = A^t(y_0 - y^*) \]

\[ y_t = A^t(y_0 - y^*) + y^* \]

Again, \( y^* = \frac{B}{1-A} \).

Then, analysis of possible magnitudes of coefficients yields Goldrég's Table 2.2 and Figure 2.3 (see handout).
\[ y_{k+1} = Ay_k + B \quad k = 0, 1, 2, \ldots \]

<table>
<thead>
<tr>
<th>Hypotheses</th>
<th>Conclusions for the sequence ( {y_k} ) is</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A \neq 1 ) &amp; ( y_0 = y^* ) &amp; ( y_k = y^* ) &amp; constant (( =y^* ))</td>
<td></td>
</tr>
<tr>
<td>( A &gt; 1 ) &amp; ( y_0 &gt; y^* ) &amp; ( y_k &gt; y^* ) &amp; monotone increasing, diverges to ( +\infty )</td>
<td></td>
</tr>
<tr>
<td>( A &gt; 1 ) &amp; ( y_0 &lt; y^* ) &amp; ( y_k &lt; y^* ) &amp; monotone decreasing, diverges to ( -\infty )</td>
<td></td>
</tr>
<tr>
<td>( 0 &lt; A &lt; 1 ) &amp; ( y_0 &gt; y^* ) &amp; ( y_k &gt; y^* ) &amp; monotone decreasing, converges to limit ( y^* )</td>
<td></td>
</tr>
<tr>
<td>( 0 &lt; A &lt; 1 ) &amp; ( y_0 &lt; y^* ) &amp; ( y_k &lt; y^* ) &amp; monotone increasing, converges to limit ( y^* )</td>
<td></td>
</tr>
<tr>
<td>( -1 &lt; A &lt; 0 ) &amp; ( y_0 \neq y^* ) &amp; ( y_k \neq y^* ) &amp; damped oscillatory, converges to limit ( y^* )</td>
<td></td>
</tr>
<tr>
<td>( A = -1 ) &amp; ( y_0 \neq y^* ) &amp; ( y_k \neq y^* ) &amp; divergent, oscillates finitely</td>
<td></td>
</tr>
<tr>
<td>( A &lt; -1 ) &amp; ( y_0 \neq y^* ) &amp; ( y_k \neq y^* ) &amp; divergent, oscillates infinitely</td>
<td></td>
</tr>
<tr>
<td>( A = 1 ) &amp; ( B = 0 ) &amp; ( y_k = y_0 ) &amp; constant (( =y_0 ))</td>
<td></td>
</tr>
<tr>
<td>( A = 1 ) &amp; ( B &gt; 0 ) &amp; ( y_k &gt; y_0 ) &amp; monotone increasing, diverges to ( +\infty )</td>
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<tr>
<td>( A = 1 ) &amp; ( B &lt; 0 ) &amp; ( y_k &lt; y_0 ) &amp; monotone decreasing, diverges to ( -\infty )</td>
<td></td>
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</table>

The various behavior types listed in Table 2.2 are roughly sketched in Figure 2.3. In each case, the graph is selected to be typical of the type of solution obtained for the specified values of \( A, B, \) and \( y_0 \). The graphs are labeled to match the rows of Table 2.2.

In the sections to follow we shall apply these results to a wide variety of difference equations which arise in the social and behavioral sciences. For purposes of easy reference, we summarize our findings in the following theorem.